The Austalian National University Semester 1, 2024
School of Computing
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COMP1130: Lambda Calculus - Worksheet 1

- **Well-formed Expressions**

  Which of the following are well defined lambda calculus expressions?

  1. λ
     Solution. No, there is no variable bound to the λ.
  2. $\lambda x.$
     Solution. No, there is no expression being abstracted by λx.
  3. $\lambda x.\lambda y.x$
     Solution. Yes.
  4. $\lambda.\lambda.y$
     Solution. No, again, λ is missing a variable.
  5. $\lambda z.z\ \lambda z.z$
     Solution. Yes.
  6. $\lambda z.(z\lambda z.)z$
     Solution. No, the λz inside has no expression.
  7. x
     Solution. Yes, variables are valid expressions.

- **Scope and Associativity**

  For the following expressions, remove as many redundant brackets as possible using the scoping laws and the property of left associativity. (Hopefully this exercise motivates why we avoid brackets where possible!)

  1. $(\lambda x.(\lambda y.(\lambda z.(zy)x)))$
     Solution. We can remove the outside parenthesis to get $\lambda x.(\lambda y.(\lambda z.(zy)x))$. Since we have left associativity, we can replace $(zy)x$ with zyx to obtain $\lambda x.(\lambda y.(\lambda z.zyx))$. Since function abstraction extends to as far right as possible, we can drop the brackets to obtain $\lambda x.\lambda y.\lambda z.zyx$.
  2. $(v((w(xy))z))$
     Solution. We can drop the outside brackets to get $v((w(xy))z)$, and use left associativity to replace $(w(xy))z$ with $w(xy)z$. This gives us $v(w(xy)z)$. We cannot remove any more brackets without changing the meaning of the expression.
  3. $(\lambda x.(\lambda y.(((\lambda z.xy)z)w)))$
     Solution. Drop outside brackets, and use the fact that the scope of λ extends to get $\lambda x.\lambda y.((\lambda z.xy)z)w$. Using left associativity, replace $((\lambda z.xy)z)w$ with $(\lambda z.xy)zw$, and therefore obtain $\lambda x.\lambda y.(\lambda z.xy)zw$. We cannot remove any more brackets without $\lambda z.$ capturing too many variables.

4. $(\lambda x.(y((\lambda z.z)w)(\lambda z.z)))$

   Solution. Drop the outside brackets, and using scoping of $\lambda x.$ to get $\lambda x.y((\lambda z.z)w)(\lambda z.z)$. Remove the last set of brackets around $\lambda z.z$ to get $\lambda x.y((\lambda z.z)w)\lambda z.z$. We cannot remove any more brackets, without the inner $\lambda z.$ capturing the w or left associativity evaluating terms in a different order.

5. $\lambda x.(\lambda y.((\lambda z.z)((xy)\lambda x.(xy)))w)$

   Solution. We can change $((xy)\lambda x.(xy))$ into $(xy\lambda x.xy)$ using left associativity and scoping rules for $\lambda x.$, which gives $\lambda x.(\lambda y.((\lambda z.z)(xy\lambda x.xy))w)$. We ave to leave the brackets on $(\lambda z.z)$ or it will capture too much, but we can remove the brackets around $((\lambda z.z)(xy\lambda x.xy))$ due to left associativity. Removing the last set of brackets around $\lambda y.$ gives $\lambda x.\lambda y.(\lambda z.z)(xy\lambda x.xy)w$.