COMP1100: Programming as Problem Solving

Slides 2: Functions and Sets

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(based on material from Ranald Clouston, Yun Kuen Cheung, and Michael Norrish)

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Semester 2 2025

Programming with Functions

Functional Programming

This course teaches a style of programming called functional programming.

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Functions

Functions are mappings between sets.

Sets

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A set is a collection of things, called its elements.

- Not in any particular order
$$\{3, 4, 8\} = \{8, 3, 4\}$$
- Each element appears only once
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Finite Sets

Some sets have only finitely many elements, so can be listed:

- A singleton, e.g. $\{3\}$ or $\{\text{True}\}$.
- The booleans $\mathbb{B}$: $\{\text{True}, \text{False}\}$

Large finite sets may not be practical to list out fully, so we use ranges:

- A set of characters: $\{\texttt{'A'}, \texttt{'B'}, \ldots, \texttt{'Z'}\}$.
- A bigger set of numbers: $\{0, 1, 2, \ldots, 100\}$.

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Infinite Sets

- The natural numbers $\mathbb{N}$: $\{0, 1, 2, \dots\}$
- The integers $\mathbb{Z}$: $\{\dots, -1, 0, 1, 2, \dots\}$
- The real numbers $\mathbb{R}$
- The strings: lists of characters of any finite length.

Combining Sets: Product

Definition

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Products can extend to more than two sets:

$$A \times B \times C \times D$$

whose members have the form (a, b, c, d).

e.g. $\mathbb{B} \times \{\text{Red}, \text{Green}, \text{Blue}\} \times \mathbb{B}$?

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Sums can extend to more than two sets:
$$A + B + C + D$$

Although we had better pick better names for tags than Left and Right!

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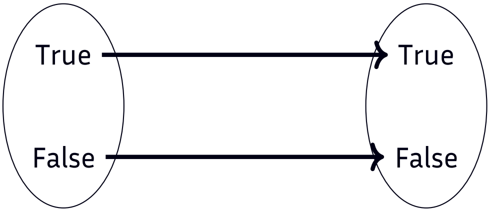

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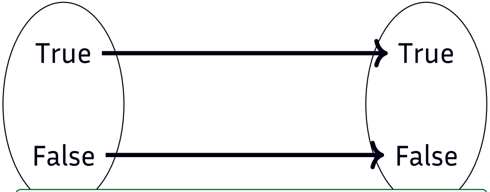

This is the identity function $\mathbb{B} \longrightarrow \mathbb{B}$.

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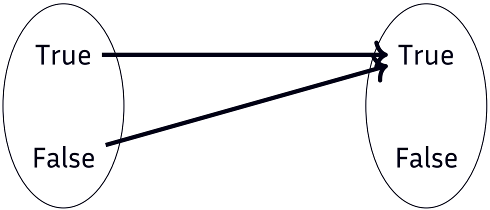

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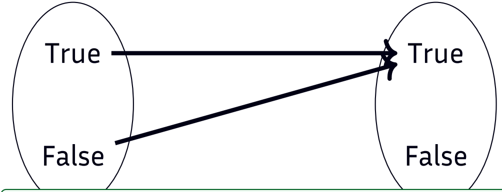

The constant function returning True.

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This is not a total function — it is partial

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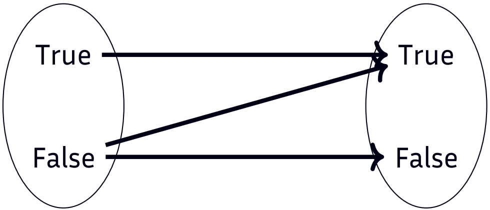

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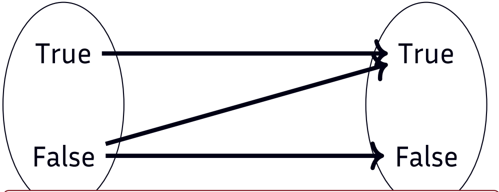

This is not a function (but a relation)

Defining Functions

Finite Domains

If the domain is finite, we can define a function by listing its output for each possible input.

For example, define:

$$
\begin{aligned}
&f :: \mathbb{B} \to \mathbb{Z} \\
&f(\mathrm{False}) = -6 \\
&f(\mathrm{True}) = 26
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Infinite (or larger finite) Domains

For larger domains, we must explain how to compute the output for all inputs:

$$\begin{aligned} &minus :: \mathbb{Z} \to \mathbb{Z} \\ &minus(x) = -x \end{aligned}$$

$$isPos :: \mathbb{Z} \to \mathbb{B}$$

$$isPos(y) = \begin{cases} \text{True} & \text{if } y \text{ is positive} \\ \text{False} & \text{if } y \text{ is negative} \\ \text{False} & \text{if } y = 0 \end{cases}$$

Here x and y are variables, which stand for any element of $\mathbb{Z}$.

Polymorphic Functions

Definition

A function that can be defined *simultaneously* for many different sets is called polymorphic.

- The identity function:

$$\begin{aligned} id &:: A \to A \\ id(a) &= a \end{aligned}$$

- The constant zero function:

$$\begin{aligned} cz &:: B \to \mathbb{Z} \\ cz(b) &= 0 \end{aligned}$$

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Examples

Let's define these together:

- $proj_L :: A \times B \to A$
- $proj_R :: A \times B \to B$
- $inj_L :: A \to A + B$
- $inj_R :: B \to A + B$

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Is this function polymorphic?

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Defined for more than one set, e.g. $\mathbb{Z}, \mathbb{R}$, but not defined for others, e.g. $\mathbb{N}, \mathbb{B}$.

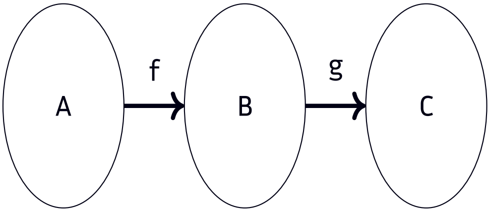
This function is polymorphic *only for sets where* $(-)$ is defined.

Combining Functions: Composition

Definition

If we have functions $f :: A \to B$ and $g :: B \to C$, we define a new function $g \circ f :: A \to C$ by

$$(g \circ f)(a) = g(f(a))$$



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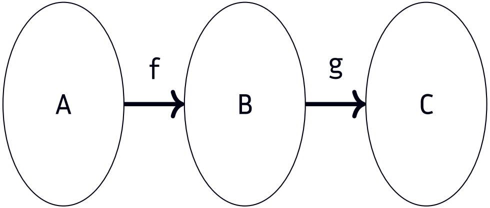

Examples

Earlier we defined $minus :: \mathbb{Z} \to \mathbb{Z}$ and $isPos :: \mathbb{Z} \to \mathbb{B}$.

- Can we define $minus \circ isPos$ or $isPos \circ minus$?

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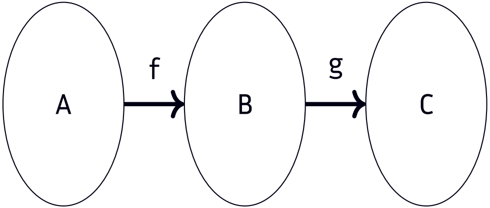

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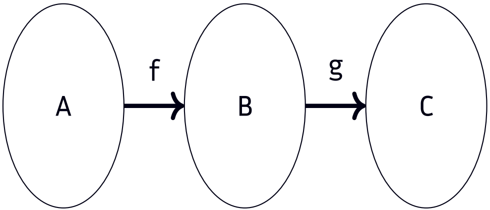

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- Can you describe in words what this combined function does?

Combining Sets: Sets of Functions

The Space of Functions

Until now we have thought of elements of sets, and functions, as different things.

But given sets A and B, we can form a new set $A \to B$ — the set of *all possible functions* from A to B.

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$$pair :: A \to (B \to A \times B)$$
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And the domain of a function can itself consist of functions:

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From Mathematics to Programming

The Space of Functions

- Syntax differs between standard mathematics and any given programming language, e.g. `f x` in Haskell instead of $f(x)$.
- Mathematicians are comfortable with infinite constructions, such as real numbers, but programmers are restricted to finite memory.
- Mathematical functions assign an output to every input — programmed functions may crash, or get stuck in a loop.
- Mathematical functions are defined entirely by their association of output to input; in programming one mathematically identical function may be better than another, e.g. it may run faster.

Haskell