COMP1600/COMP6260: Mid-semester exam (solutions)

2021-2

Q1: Propositional Logic

Multiple choice question (Wattle).

Version 1:

Use the new connective (◆) defined using the truth table below to answer parts (1) to (6):

p	q	$p \blacklozenge q$
F	F	F
F	T	F
T	F	T
T	T	F

(For parts (1) to (6), there might be more than one true answer. You need to select only one of them.)

For each of the formulae in (1) to (3) below, select:

- 'tautology', if the formula is a tautology,
- 'contradiction', if the formula is a contradiction, or
- a situation in which the formula evaluates to False, if it's a contingency.

1. (p ◆ q) ∧ (q ◆ p) Contradiction.
2. (p → q) ◆ q Contingency, and evaluates to F for p = T, q = F.
3. (p ◆ q) → (p ∧ ¬q) Tautology.

For parts (4) to (6), select a correct choice:

4. (p◆q) can be expressed using only connectives from which of the following sets: {→, F}
5. Which one of the following sets is not expressively complete: {◆, F}
6. Which one of the following sets is expressively complete: {◆,T}

Version 2:

Use the new connective (◆) defined using the truth table below to answer parts (1) to (6):

p	q	$p \blacklozenge q$
F	F	T
F	T	F
T	F	T
T	T	T

(For parts (1) to (6), there might be more than one true answer. You need to select only one of them.)

For each of the formulae in (1) to (3) below, select:

- 'tautology', if the formula is a tautology,
- 'contradiction', if the formula is a contradiction, or
- a situation in which the formula evaluates to False, if the formula is a contingency.

1. (p ◆ q) ∧ (q ◆ p) [Contingency, and evaluates to F for p = T, q = F.]
2. (p → q) ◆ q [Tautology.]
3. (p ◆ q) → (p ∧ ¬q) [Contingency, and evaluates to F for p = F, q = F.]

For parts (4) to (6), select a correct choice:

4. (p ◆ q) can be expressed using only connectives from which of the following sets: [{→, F}]
5. Which one of the following sets is not expressively complete: [{◆,T}]
6. Which one of the following sets is expressively complete: [{◆, F}]

Q2: Propositional Logic + Natural Deduction

Determine, and explicitly mention, if the formula below is a tautology, contradiction, or contingency

$$(a \to (b \to c)) \to (a \land b \to c).$$

- If tautology, give a proof in Natural Deduction. Use only the rules given in this Appendix 1 and justify every step of your proof;
- If contradiction, give a proof in Natural Deduction of its negation. Use only the rules given in this Appendix 1 and justify every step of your proof;
- If contingency, give two truth-assignments to its variables, one that makes the formula false, and one that makes it true.

Solution: It is a tautology

1	$\quad a \to (b \to c)$	
2	$\qquad a \land b$	
3	$\qquad a$	$\land$-E, 2
4	$\qquad b \to c$	$\to$-E, 3, 1
5	$\qquad b$	$\land$-E, 2
6	$\qquad c$	$\to$-E, 5, 4
7	$\quad a \land b \to c$	$\to$-I, 2–6
8	$(a \to (b \to c)) \to (a \land b \to c)$	$\to$-I, 1–7

Q3: First Order Logic + Natural Deduction

Show that the rule

$$\frac{\neg\forall x.\exists y.\neg P(x,y)}{\exists x.\forall y.P(x,y)}$$

is derivable in natural deduction.

Use only the rules given in this Appendix 1 and justify every step of your proof.

Hint: you might want to use proof by contradiction repeatedly.

Solution: There are several ways. Here are 3 of them:

Solution 1:

1	$\mid\ \neg\, \forall x. \exists y. \neg\, P(x,y)$	
2	$\mid\ \mid\ \neg\, \exists x. \forall y. P(x,y)$	
3	$\mid\ \mid\ a \mid\ \mid\ \neg\, \exists y. \neg\, P(a,y)$	
4	$\mid\ \mid\ \ \mid\ \mid\ b \mid\ \mid\ \neg\, P(a,b)$	
5	$\mid\ \mid\ \ \mid\ \mid\ \ \mid\ \mid\ \exists y. \neg\, P(a,y)$	∃-I, 4
6	$\mid\ \mid\ \ \mid\ \mid\ \ \mid\ \mid\ \bot$	¬I, 5, 3
7	$\mid\ \mid\ \ \mid\ \mid\ \ \mid\ P(a,b)$	PC, 4–6
8	$\mid\ \mid\ \ \mid\ \mid\ \forall y. P(a,y)$	∀-I, 7
9	$\mid\ \mid\ \ \mid\ \mid\ \exists x. \forall y. P(x,y)$	∃-I, 8
10	$\mid\ \mid\ \ \mid\ \mid\ \bot$	¬I, 9, 2
11	$\mid\ \mid\ \ \mid\ \exists y. \neg\, P(a,y)$	PC, 3–10
12	$\mid\ \mid\ \forall x. \exists y. \neg\, P(x,y)$	∀-I, 11
13	$\mid\ \mid\ \bot$	¬I, 12, 1
14	$\mid\ \exists x. \forall y. P(x,y)$	PC, 2–13

Solution 2: We first prove $\exists x.\neg\,\exists y.\neg\, P(x,y)$ as an intermediate step.

1	$\neg\,\forall x.\exists y.\neg\, P(x,y)$	
2	$\quad \neg\,\exists x.\neg\,\exists y.\neg\, P(x,y)$	
3	$\quad a \quad\quad \neg\,\exists y.\neg\, P(a,y)$	
4	$\quad\quad\quad \exists x.\neg\,\exists y.P(x,y)$	$\exists$-I, 3
5	$\quad\quad\quad \bot$	$\neg$I, 4, 2
6	$\quad\quad \exists y.\neg\, P(a,y)$	PC, 3–5
7	$\quad \forall x.\exists y.\neg\, P(x,y)$	$\forall$-I, 6
8	$\quad \bot$	$\neg$I, 7, 1
9	$\exists x.\neg\,\exists y.\neg\, P(x,y)$	PC, 2–8
10	$a \quad \neg\,\exists y.\neg\, P(a,y)$	
11	$\quad b \quad\quad \neg\, P(a,b)$	
12	$\quad\quad\quad \exists y.\neg\, P(a,y)$	$\exists$-I, 11
13	$\quad\quad\quad \bot$	$\neg$I, 12, 10
14	$\quad\quad P(a,b)$	PC, 11–13
15	$\quad \forall y.P(a,y)$	$\forall$-I, 14
16	$\quad \exists x.\forall y.P(x,y)$	$\exists$-I, 15
17	$\exists x.\forall y.P(x,y)$	$\exists$-E, 9, 10–16

Solution 3: We first prove the following lemmas:

$$\frac{\neg\,\forall x.P(x)}{\exists x.\neg\,P(x)}\,(\text{L1})$$

1	$\neg\,\forall x.P(x)$	
2	$\quad \neg\,\exists x.\neg\,P(x)$	
3	$\quad a \quad\quad \neg\,P(a)$	
4	$\quad\quad\quad\quad \exists x.\neg\,P(x)$	∃-I, 3
5	$\quad\quad\quad\quad \bot$	¬I, 4, 2
6	$\quad\quad\quad P(a)$	PC, 3–5
7	$\quad\quad \forall x.P(x)$	∀-I, 6
8	$\quad\quad \bot$	¬I, 7, 1
9	$\exists x.\neg\,P(x)$	PC, 2–8

and

$$\frac{\neg\,\exists x.\neg\,P(x)}{\forall x.P(x)}\,(\text{L2})$$

1	$\neg\,\exists x.\neg\,P(x)$	
2	$a \quad\quad \neg\,P(a)$	
3	$\quad\quad\quad \exists x.\neg\,P(x)$	∃-I, 2
4	$\quad\quad\quad \bot$	¬I, 3, 1
5	$\quad\quad P(a)$	PC, 2–4
6	$\forall x.P(x)$	∀-I, 5

from which the result then easily follows

1	$\neg\,\forall x.\exists y.\neg\,P(x,y)$	
2	$\exists x.\neg\,\exists y.\neg\,P(x,y)$	L1, 1
3	$a \quad \neg\,\exists y.\neg\,P(a,y)$	
4	$\quad\quad \forall y.P(a,y)$	L2, 3
5	$\quad\quad \exists x.\forall y.P(x,y)$	∃-I, 4
6	$\exists x.\forall y.P(x,y)$	∃-E, 2, 3–5

Q4: Structural Induction

Consider the following two functions:

```
treeSize :: Tree a -> Int
treeSize Null = 0                                     --TS1
treeSize (Node l x r) = 1 + (treeSize l) + (treeSize r)   --TS2
```

and

```
bloom :: Tree a -> a -> Tree a
bloom Null flower = Node Null flower Null                          --B1
bloom (Node l x r) flower = Node (bloom l flower) x (bloom r flower)  --B2
```

The goal of this exercise is to show that

$$\forall\, t.\, \forall\, f. \mathit{treeSize}(\mathit{bloomtf}) = 2 * (\mathit{treeSizet}) + 1.$$

1. What precisely should we prove by induction? State a property P, including possible quantifiers, so that proving this property by induction implies the (above) goal of this exercise.
2. State (including possible quantifiers) and prove the base case goal.
3. State (including possible quantifiers) the inductive hypotheses of the proof.
4. State (including possible quantifiers) and prove the step case goal.

Solution:
We prove by tree induction on `t`.

1. P(t) = ∀ `f.  treeSize (bloom t f) = 2 * (treeSize t) + 1`
2. Base case, we need to prove that

   ∀ `f.  treeSize (bloom Nul f) = 2 * (treeSize Nul) + 1`

   Or `P(Nul)`

```
LHS = treeSize (bloom Null f)
    = treeSize (Node Null f Null) --B1
    = 1 + treeSize Null + treeSize Null --TS2
    = 1 + 2 * treeSize Null             --maths
    = 2 * (treeSize Null) + 1           --maths
= RHS
```

3. Inductive Hypotheses:

   ∀ f. `treeSize (bloom l f) = 2 * (treeSize l) + 1 --IH1`
   and
   ∀ f. `treeSize (bloom r f) = 2 * (treeSize r) + 1 --IH2`

4. Step case goal : ∀ `x`, ∀ `f.treeSize (bloom (Node t1 x t2) f) =`
   `2 * (treeSize (Node t1 x t2)) + 1`

   Or ∀ `x P(Node l x r)`

```
treeSize (bloom (Node l x r) f)
= treeSize (Node (bloom l f) x (bloom r f))           --B2
= 1 + treeSize (bloom l f) + treeSize (bloom r f)     --TS2
= 1 + (2 * (treeSize l) + 1) + (2 * (treeSize r) + 1) --IH1,IH2
= 2 *(1 + treeSize l + treeSize r) + 1                --maths
= 2 * treeSize (Node l x r) + 1                       --TS2
```

Q5: Hoare Logic Semantics

Multiple choice question (Wattle).

Consider the following Hoare triple with unspecified precondition *P*, and unspecified postcondition *Q* (all variables are of type integer):

{*P*}
while $x \neq y$ do
    if x > y then
        x := x-y
    else
        y := y-x
{*Q*}

For (1) to (4), consider the proposed precondition *P* and postcondition *Q*, and select the correct statement (no proofs are required):

1. P ≡ x=y, Q ≡ x>0 ∧ y>0: The Hoare triple is invalid because for negatives values of x that make P true, Q cannot be established.
2. P ≡ true, Q ≡ x=1 ∧ y=1: The Hoare triple is invalid because for x=4 and y=6, the program terminates, but Q is not stablished.
3. P ≡ x > 0 ∧ y > 0, Q ≡ x+y >= 2: The Hoare triple is valid.
4. P ≡ x+y > 0, Q ≡ false: The Hoare triple is invalid because for x = 4 and y = 3, the program terminates.

Q6: Hoare Logic (proof)

Version 1: Consider the following Hoare Triple (all variables are of type integer):

```
{true}
if g > e + 5 then g := g-e; e := -g
else e := (g-e)-5; g := 12
{g > 0 and e <= 0}
```

Give a Hoare-Logic Proof of this triple. You may only use the rules given in this Appendix 2. Be sure to justify every step of your proof.

solution:

```
1. {g-e>0 and -(g-e)<=0} g:=g-e {g > 0 and -g <= 0} (Assignment)
2. true and g > e+5 -> g-e>0 and -(g-e)<=0 (Logic)
Nothing that:
     g-e > 0 <=> g > e
and that
     -(g-e) <= 0 <=> -g+e <= 0 <=> g >= e
3. {true and g > e+5} g:=g-e {g > 0 and -g <= 0} (Prec. Str., 1, 2)
4. {g > 0 and -g <= 0} e:= -g {g > 0 and e <= 0} (Assignment)
5. {true and g > e+5} g := g-e; e := -g {g > 0 and e <= 0} (Seq., 3, 4)

6. {12 > 0 and (g-e)-5 <= 0} e := (g-e)-5 {12 > 0 and e <= 0} (Assignment)
7. true and !(g > e+5) -> 12 > 0 and (g-e)-5 <= 0 (Logic)
Noting that
     !(g > e+5) <=> g <= e+5 <=> g-e-5 <= 0
8. {true and !(g > e+5)} e := (g-e)-5 {12 > 0 and e <= 0} (Prec. Str., 6,7)
9. {12 > 0 and e <= 0} g := 12  {g > 0 and e <= 0} (Assignment)
10. {true and !(g > e+5)}  e := (g-e)-5; g := 12 {g > 0 and e <= 0} (Seq., 8,9)

11. {true}
            if g > e+5
            then g := g-e; e := -g
            else e := (g-e)-5; g := 12
    {g > 0 and e <= 0} (Conditional, 5,10)
```

Version 1: Consider the following Hoare Triple (all variables are of type integer):

```
{true}
if u > b + 3 then
    u := u-b;
    b := -u
else
    b := (u-b)-3;
    u := 10
{u > 0 and b <=0 }
```

Give a Hoare-Logic Proof of this triple. You may only use the rules given in this Appendix 2. Be sure to justify every step of your proof.

solution:

```
1. {u > 0 and -u <= 0} b := -u {u > 0 and b <= 0}  (Assignment)
2. {u-b > 0 and -(u-b) <= 0} u := u-b {u > 0 and -u <= 0}  (Assignment)
3. true and u > b + 3 -> u-b > 0 and -(u-b) <= 0 (Logic)
Noting that
     u-b > 0 <=> u > b
and that
     -(u-b) <= 0 <=> -u+b <= 0 <=> u >= b
4. {true and u > b + 3} u := u-b {u > 0 and -u <= 0}  (Prec. Str., 2, 3)
5. {true and u > b + 3} u := u-b; b := -u {u > 0 and b <= 0}  (Seq., 4, 1)


6. {10 > 0 and b <= 0} u := 10 {u > 0 and b <= 0} (Assignment)
7. {10 > 0 and (u-b)-3 <= 0} b := (u-b)-3 {10 > 0 and b <= 0} (Assignment)
8. true and !(u > b + 3) -> 10 > 0 and (u-b)-3 <= 0 (Logic)
Noting that
     !(u > b + 3) <=> u <= b + 3 <=> u-b-3 <= 0
9. {true and !(u > b + 3)} b := (u-b)-3 {10 > 0 and b <= 0} (Prec. Str., 7, 8)
10. {true and !(u > b + 3)} b := (u-b)-3; u := 10 {u > 0 and b <= 0} (Seq., 9, 6)

11. {true}
        if u > b + 3
        then u := u-b; b := -u
        else b := (u-b)-3; u := 10
    {u > 0 and b <= 0}  (Conditional, 5,10)
```

1 Appendix: Natural Deduction Rules

Propositional Calculus

$(\wedge I)\quad \dfrac{p \qquad q}{p \wedge q}$

$(\wedge E)\quad \dfrac{p \wedge q}{p} \qquad \dfrac{p \wedge q}{q}$

$(\vee I)\quad \dfrac{p}{p \vee q} \qquad \dfrac{p}{q \vee p}$

$$(\vee E)\quad \dfrac{p \vee q \qquad \begin{matrix}[p] \\ \vdots \\ r\end{matrix} \qquad \begin{matrix}[q] \\ \vdots \\ r\end{matrix}}{r}$$

$$(\to I)\quad \dfrac{\begin{matrix}[p] \\ \vdots \\ q\end{matrix}}{p \to q}$$

$(\to E)\quad \dfrac{p \qquad p \to q}{q}$

$$(\neg I)\quad \dfrac{\begin{matrix}[p] \\ \vdots \\ F\end{matrix}}{\neg p}$$

$(\neg E)\quad \dfrac{p \qquad \neg p}{F}$

$$(\mathsf{PC})\quad \dfrac{\begin{matrix}[\neg p] \\ \vdots \\ F\end{matrix}}{p}$$

$(T)\quad \dfrac{}{T}$

Predicate Calculus

$(\forall I)\quad \dfrac{P(a) \qquad (a \text{ arbitrary})}{\forall x.\ P(x)}$

$(\forall E)\quad \dfrac{\forall x.\ P(x)}{P(a)}$

$$(\exists I) \qquad \frac{P(a)}{\exists x P(x)}$$

$$(\exists E) \qquad \frac{\exists x P(x) \qquad \begin{matrix} [P(a)] \\ \vdots \\ q \end{matrix} \qquad (a \text{ arbitrary})}{q \quad (a \text{ is not free in } q)}$$

2 Hoare Logic Rules (partial correctness)

- Assignment:

$$\{Q(e)\}\ \texttt{x := e}\ \{Q(x)\}$$

- Precondition Strengthening:

$$\frac{P_s \ \rightarrow\ P_w \qquad \{P_w\}\ S\ \{Q\}}{\{P_s\}\ S\ \{Q\}}$$

  You can always replace predicates by equivalent predicates, i.e. if $P_s \leftrightarrow P_w$; just label your proof step with 'precondition equivalence'.

- Postcondition Weakening:

$$\frac{\{P\}\ S\ \{Q_s\} \qquad Q_s\ \rightarrow\ Q_w}{\{P\}\ S\ \{Q_w\}}$$

  You can always replace predicates by equivalent predicates, i.e. if $Q_s \leftrightarrow Q_w$; just label your proof step with 'postcondition equivalence'.

- Sequence:

$$\frac{\{P\}\ S_1\ \{Q\} \qquad \{Q\}\ S_2\ \{R\}}{\{P\}\ S_1;\ S_2\ \{R\}}$$

- Conditional:

$$\frac{\{P \wedge b\}\ S_1\ \{Q\} \qquad \{P \wedge \neg\, b\}\ S_2\ \{Q\}}{\{P\}\ \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \{Q\}}$$