Australian National University

COMP1730/COMP6730
Programming for Scientists

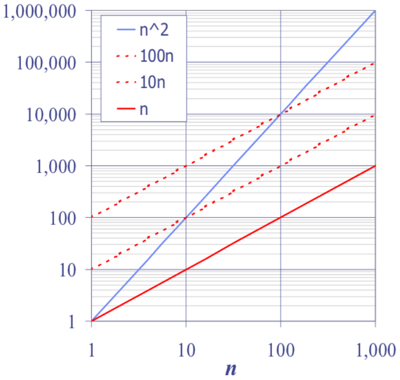
Algorithm and problem complexity

Algorithm complexity

* The time (memory) consumed by an algorithm:
  - Counting "elementary operations" (not μs).
  - Expressed as a function of the size of its arguments.
  - In the worst case.
* Complexity describes scaling behaviour: How much does runtime grow if the size of the arguments grow by a certain factor?
  - Understanding algorithm complexity is important when (but usually only when) dealing with large problems.

Big-O notation

- $O(f(n))$ means roughly "a function that grows at the rate of $f(n)$, for large enough n".
- For example,
  - $n^2 + 2n$ is $O(n^2)$
  - $100n$ is $O(n)$
  - 10^{12} is $O(1)$.



(Image by Lexing Xie)

Example

- Find the greatest element $\leq x$ in an *unsorted* sequence of *n* elements. (For simplicity, assume some element $\leq x$ is in the sequence.)
- Two approaches:

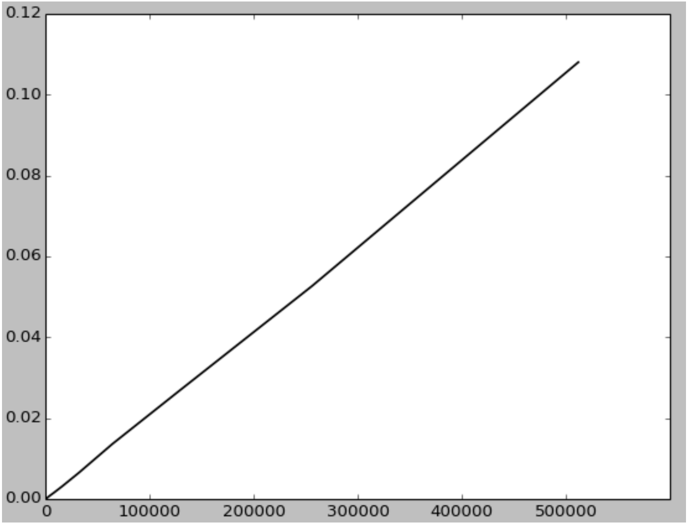
a) Search through the sequence; or
b) First sort the sequence, then find the greatest element $\leq x$ in a *sorted* sequence.

Searching an unsorted sequence

```
def unsorted_find(x, ulist):
    best = min(ulist)
    for elem in ulist:
        if elem == x:
            return elem
        elif elem <= x:
            if elem > best:
                best = elem
    return best
```

Analysis

* Elementary operation: comparison.
  - Can be arbitrarily complex.
* If we're lucky, `ulist[0] == x`.
* Worst case?
  - `ulist = [0, 1, 2, ..., x - 1]`
  - Compare each element with `x` and current value of `best`
* What about `min(ulist)`?
* $f(n) = 2n$, so $O(n)$



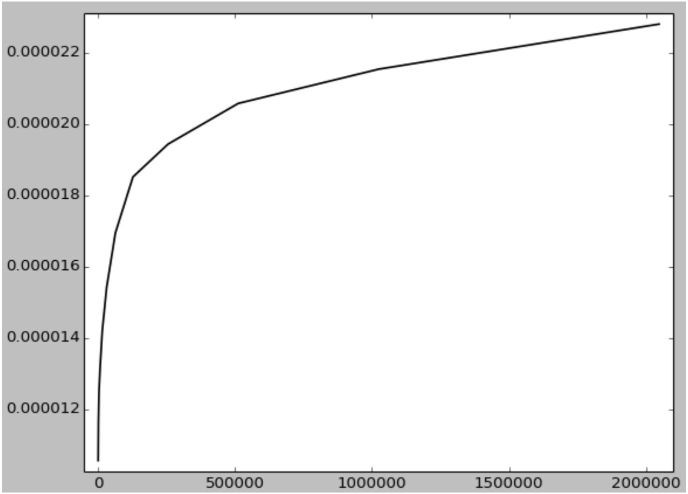
Measured runtime

Searching a sorted sequence

```
def sorted_find(x, slist):
    if slist[-1] <= x:
        return slist[-1]
    lower = 0
    upper = len(slist) - 1
    while (upper - lower) > 1:
        middle = (lower + upper) // 2
        if slist[middle] <= x:
            lower = middle
        else:
            upper = middle
    return slist[lower]
```

Analysis

- Loop invariant: `slist[lower] <= x` and `x < slist[upper]`.
- How many iterations of the loop?
  - Initially, `upper - lower` $= n - 1$.
  - The difference is halved in every iteration.
  - Can halve it at most $\log_2(n)$ times before it becomes 1.
- $f(n) = \log_2(n) + 1$, so $O(\log(n))$.



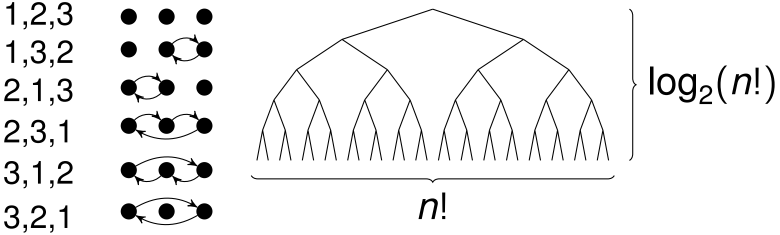
Measured runtime

Problem complexity

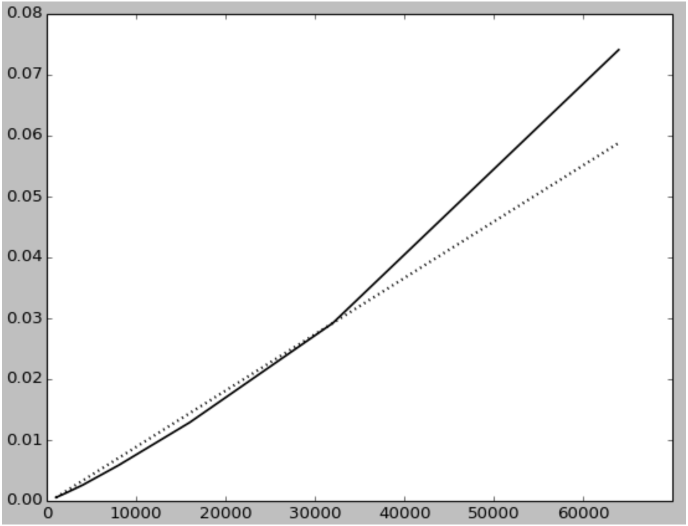
* The complexity of a problem is the time (memory) that *any* algorithm *must* use, in the worst case, to solve the problem, as a function of the size of the arguments.
* The hierarchy theorem: For any computable function $f(n)$ there is a problem that requires time greater than $f(n)$. (Analogous result for memory.)

How fast can you sort?

* Any sorting algorithm that uses only pair-wise comparisons needs $n \log(n)$ comparisons in the worst case.



* $\log_2(n!) \geq n \log(n)$ for large enough n.



Measured runtime (`list.sort`)

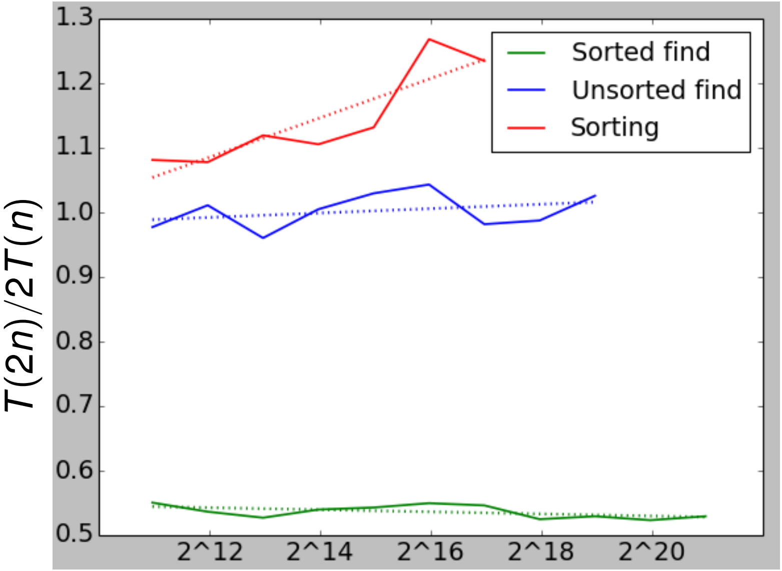
Points of comparison

- Algorithm (a): $O(n)$
- Algorithm (b): $n\log(n) + \log(n) = O(n\log(n))$

	$n = 64$k	$n = 128$k	$n = 512k$
Unsorted find	0.013 s	0.026 s	0.108 s
Sorted find	0.000017s	0.000018s	0.00002 s
Sorting	0.07 s	0.18 s	

Rate of growth

- Algorithm uses $T(n)$ time on input of size n.
- If we double the size of the input, by what factor does the runtime increase?


T(2n)/2T(n)
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
2^12
2^14
2^16
2^18
2^20
Sorted find
Unsorted find
Sorting

Caution

* "Premature optimisation is the root of all evil in programming."

  – C.A.R. Hoare

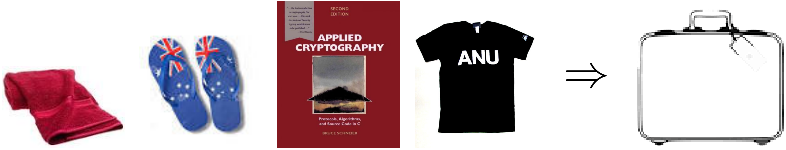
* Remember: Scaling behaviour becomes important when (and *only* when) problems become *large*, or when they need to be solved a *many times*.

NP-Completeness

Example

* The subset sum problem: Given n integers $w_1, \ldots, w_n$, is there a subset of them that sums to exactly C?

  (Also known as the "(exact) knapsack problem":



$w_0 = 5$ $w_1 = 2$ $w_2 = 9$ $w_3 = 1$ $C = 16$.)

```
def subset_sum(w, C):
    if len(w) == 0:
        return C == 0
    # including w[0]
    if w[0] <= C:
        if subset_sum(w[1:], C - w[0]):
            return True
    # excluding w[0]
    if subset_sum(w[1:], C):
        return True
    return False
```

Analysis

* Count recursive function calls (no loops, so every call does a constant max amount of work).
* Assume argument size (n) is number of weights.
* Worst case?
  - If the answer is `False` and C is less than but close to $\sum_i w_i$, almost every call makes two recursive calls.
* $f(n+1) = 2f(n)$, $f(0) = 1$ means that $f(n) = 2^n$.

Finding vs. checking an answer

- Sorting a list vs. $O(n\log(n))$
  checking if it's already sorted $O(n)$

- Finding a subset of $w_1, \ldots, w_n$ $O(2^n)$
  that sums to C vs.
  checking if a sum is equal to C $O(n)$

NP-complete problems

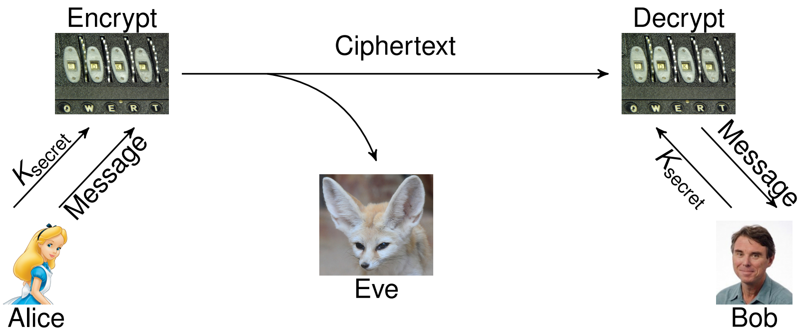
* A problem is **in NP** iff there is an answer-checking algorithm that runs in polynomial time ($O(n^c)$, c constant).
* NP stands for **N**on-deterministic **P**olynomial time.
* A problem is **NP-complete** if it's in NP and *at least as hard as every other problem in NP*.
* We think there is no polynomial time algorithm for solving NP-complete problems, but *we don't know*.

There are many NP-complete problems...

- Most populous intractable problem class.
  - Solving a system of *integer* linear equations.
  - The Knapsack problem.
- `http://www.nada.kth.se/~viggo/wwwcompendium/wwwcompendium.html` lists over 700 NP-complete optimisation problems.
- The question of whether there is a polynomial time algorithm for solving NP-complete problems is one of the most significant unresolved questions in computer science. There is a $US1,000,000 prize for an answer.

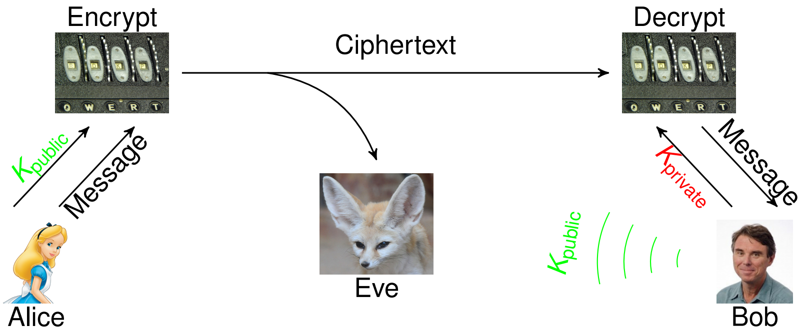
Why Complexity is (Sometimes) a Good Thing

Cryptographic Characters



* Eve can intercept the ciphertext, but without knowing K_{secret} can’t read the message.
* Alice and Bob must agree on K_{secret}.

Public Key Cryptography



- K_{public} can only be used to encrypt.
- Decrypting with $K_{private}$ is easy, but decrypting without knowing $K_{private}$ is *(NP-)hard*.

Example: Proof of Identity

* Alice is chatting with "Bob" on-line, but wants to be sure it's really Bob.

1. Alice picks a random number N and sends $C = \text{Encrypt}(K_{\text{public}}, N)$ to "Bob".
2. Bob *quickly* computes $N = \text{Decrypt}(K_{\text{private}}, C)$ and sends N back to Alice.

Repeat **1–2** many times to make sure "Bob" didn't make a lucky guess.
Succeeding every time proves he knows K_{private}, which we assume only Bob does.