The Australian National University Semester 1, 2025
School of Computing **Final Exam**

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Logic (COMP2620)

PLEASE READ THIS BEFORE YOU START WORKING ON YOUR EXAM:

- You have **15 minutes** of reading time and an additional **180 minutes** to solve the exercises. The total score is 100 marks.
- This exam is split into **7** topics, each of which is split into questions. Topics and questions are worth varying numbers of marks, and are labelled with how many marks they are worth. Partial marks will be available for all questions.
- Your answers should go into the provided booklet. Your booklet should be clearly labelled with your University ID, and not your name. Each answer should be clearly labelled by which topic and question it is for, e.g. 2.3. If your answer to a question is spread across non-consecutive pages, please clearly indicate this. If you need another booklet, you may request one.
- You may use scribble paper, and may ask for more. However, note that the scribble papers are NOT marked.
- This exam will contribute 50% to your overall grade for the course.

Exercise 1 **Truth Tables** 7 marks

1. **(3 marks)** Construct the truth table for the propositions
   - $(p \to q) \to r$
   - $p \to (q \to r)$

   Clearly indicate which columns are your final answer for each proposition. Give a brief explanation in English in which circumstances these propositions have different truth values.

2. **(4 marks)** Construct the truth table for the sequent

$$p \vee q \to q \vee r,\ q \to \neg p,\ r \to q \to \bot \vdash \neg r$$

   Clearly indicate which columns are your final answer for each proposition. Which single propositional variable could be added to the premises to make this a valid argument?

Exercise 2 **Natural Deduction** 16 marks

Prove the following sequents using natural deduction. You should use the five part notation for natural deduction: which premises are being used; a line number; a proposition or first order logic formula; previous lines used; and the rule name.

1. **(5 marks)** $(p \to q) \to (q \to p) \vdash q \to p$
2. **(5 marks)** $\exists x \forall y\, Rxy, \neg Rcc \vdash \exists x(\neg x = c \wedge Rxx)$, where c is a constant and R is a binary predicate.
3. **(6 marks)** $\forall x(Px \vee Q) \vdash \neg\exists x \neg Px \vee Q$, where P is a unary predicate and Q is a nullary predicate.

Exercise 3 **Soundness and Completeness** 14 marks

1. **(3 marks)** Suppose that the following rule was added to our natural deduction rules for propositional logic:

$$\frac{\Gamma \vdash \varphi \quad \Gamma', \varphi \vdash \psi}{\Gamma, \Gamma' \vdash \psi}$$

   Use induction to show that this rule is sound with respect to propositional logic semantics (truth tables). Be careful to clearly state your induction hypothesis.

2. **(3 marks)** Suppose that we have a proposition φ containing one variable p, and that φ has value 1 on both rows of its truth table. Restate the 'main lemma' of the completeness proof for this case (for example, you should replace π_k with an actual proposition). Then give a clear English language explanation of why we can hence conclude completeness for the particular case of the sequent $\vdash \varphi$.

   (Recall that the main lemma states

   - if φ is 1 in the k'th row of its truth table, then $\pi_k \vdash \varphi$ can be proved by natural deduction;
   - If φ is 0 in the k'th row of its truth table, then $\pi_k \vdash \neg\varphi$ can be proved by natural deduction.

   for any φ, where π_k is the 'special proposition' corresponding to the k'th row of the truth table.)

3. **(4 marks)** Suppose that the following rule was added to our natural deduction rules for first order logic:

$$\frac{\Gamma \vdash \forall x\, \varphi}{\Gamma \vdash \exists x\, \varphi}$$

   Use induction to show that this rule is sound with respect to first order logic semantics. Be careful to clearly state your induction hypothesis.

4. **(4 marks)** The substitution lemma states that

$$\vDash_{\mathcal{M},e} \varphi[t/x] \text{ if and only if } \vDash_{\mathcal{M},e[x \mapsto t^{\mathcal{M},e}]} \varphi$$

   Suppose that substitution for $\forall$-formulas was *wrongly* implemented as

$$(\forall x \psi)[t/x] \;=\; \forall x(\psi[t/x])$$

   with no side conditions. Present a first order logic signature, a model for that signature, an environment (if necessary), a formula in place of φ, and a term in place of t, which together show that, if substitution were implemented in this way, the substitution lemma would not be correct. Give a mathematical argument for why this is a counterexample.

Exercise 4 **Modelling from English** 15 marks

1. **(4 marks)** Consider the following rules for a course with two assignments which everyone sits, and an extra assignment which only some students are offered. Each assignment is marked as pass or fail. Convert each rule into propositional logic (You may use standard notation - $\wedge, \vee$, etc. - or Logic4Fun notation - AND, OR, etc.). Be explicit about which propositional variables you are using, and what their English language meaning is.

   - If you pass both assignment 1 and assignment 2, then you are not offered the extra assignment, but simply pass the course.
   - If you passed one of the assignments, but not the other one, then you will be offered the extra assignment. Note that other students might, or might not, be offered the extra assignment also.
   - If you are offered the extra assignment then you pass the course if and only if you pass the extra assignment.

   What further proposition could be added to the list that carries the meaning 'there is no other way to pass the course'?

2. **(5 marks)** Translate the following paragraph into either a series of first order logic formulas, or Logic4Fun notation. Be consistent in which notation you use. If using first order logic notation, be explicit about which functions or predicates you are using, and what their English language meaning is. If using Logic4Fun notation, be explicit about what needs to be put into the Sorts and Vocabulary boxes, and what the English language meaning of each line is, before giving your Constraints.

   "Arguments can have any number of premises, but only one conclusion. If all the premises are true, and the argument is valid, then the conclusion is true. A philosopher showed me two arguments; let's call them X and Y. Both X and Y were valid, but only one had a true conclusion. Therefore one of their premises cannot be the conclusion of any valid argument whose premises are all true"

3. **(6 marks)** On the the weekend you need to drive from your home in the Inner North of Canberra to do errands in Belconnen, Gungahlin, and South Canberra, then return to the Inner North. You may perform these errands in any order. To be efficient, you should avoid visiting anywhere other than these four areas, and avoid visiting any of the four areas more than once, other than starting and finishing in the Inner North. You can drive from anywhere to anywhere, except that to get between Gungahlin and South Canberra you need to go through either Belconnen or the Inner North in between.

   Write an LTL specification that your driving plan must obey. Consider both the conditions above and whether there are any 'commonsense' conditions that need to be specified to avoid impossible plans. Use the proposition variables i (you are in the Inner North), b (Belconnen), g (Gungahlin), and s (South Canberra).

Exercise 5 **Tableaux** 23 marks

For all tableaux in this section, you should number your lines; label on the right each new signed proposition or formula by which lines justify it; and cross each branch that can close, with line justifications beside any crosses. If a tableau becomes repetitive in parts you may leaves the later lines out, as long as you write a clear English language explanation of what is happening. If you find any terminated open branch, you do not need to explore any other branch.

1. **(5 marks)** Use the tableaux method to prove valid the sequent

$$p \vee q,\ q \to r,\ \neg(r \vee \bot) \vdash p$$

2. **(4 marks)** Use the tableaux method to prove the satisfiability of the pair of signed propositions

   - **T** : $r \vee s$
   - **F** : $p \to q \to r \wedge s$

   Explicitly state an assignment of truth values to propositional variables satisfying these signed propositions.

3. **(5 marks)** Use the tableaux method to determine whether the following sequent is valid.

$$\exists x(Px \to Qx),\ \exists x Px \vdash \exists x Qx$$

   where P and Q are unary predicates. State in English whether the sequent is valid or not. If it is not valid, you do *not* need to explicitly extract a counterexample.

4. **(5 marks)** Use the tableaux method to determine whether the following pair of signed propositions are satisfiable.

   - **T** : Fp
   - **T** : $G(p \to XG\neg p)$

   State in English whether the signed propositions are satisifable or not. If they are satisfiable, extract and draw a satisfying model.

5. **(4 marks)** Use the tableaux method to determine whether the following pair of signed propositions are satisfiable.

   - **T** : $G(\neg p\, U\, q)$
   - **T** : $G(\neg q\, U\, p)$

   State in English whether the signed propositions are satisifable or not. If they are satisfiable, extract and draw a satisfying model.

Exercise 6 **Semantics** 12 marks

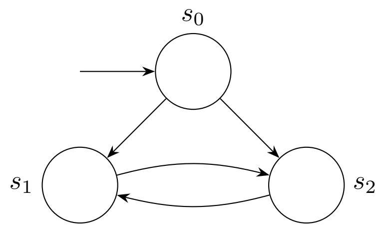
1. **(2 marks)** Consider the first order logic signature with a unary function f and binary function g. Consider a model for this signature with universe of discourse the natural numbers $0, 1, 2, \ldots$, with f interpreted as the square of its input (multiplication of the input by itself) and with g interpreted as addition.

   Define an *environment* for this model so that the formula

   $$\exists z(g(fx, fy) = fz)$$

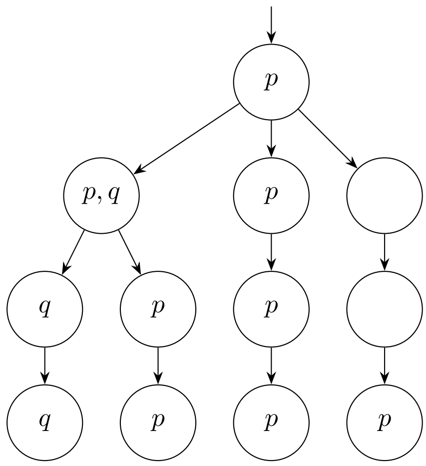
   is satisfied. Do not define the environment's action on any variable not appearing free in the formula.

2. **(2 marks)** Consider the transition system



   Suppose that this system should be able to terminate if it is in state s_1 or s_2, but not if it is in s_0. Draw a new transition system that is the same as the system above, except that it also has this property.

3. **(2 marks)** Consider the computation tree



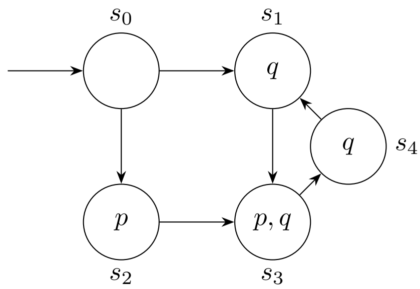
   Suppose each path through the tree continues, below the lowest level drawn, without branching or changing how propositions are labelled. Draw a transition system that has this computation tree.

4. **(3 marks)** Prove using the semantics of LTL (*not* using tableaux) that $GX\varphi$ is equivalent to $XG\varphi$ for any proposition φ.

5. **(3 marks)** Prove using the semantics of CTL* that $A[\varphi]$ is equivalent to $E[A[\varphi]]$ for any path proposition φ.

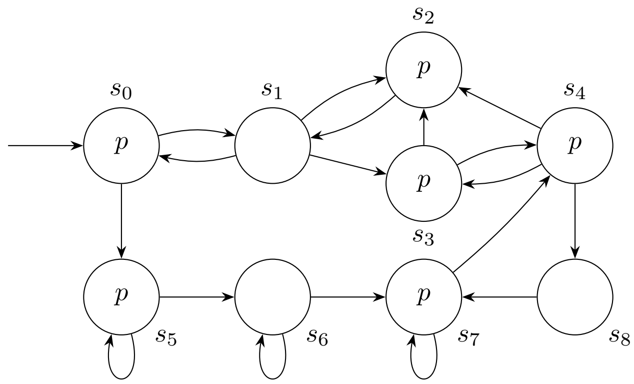
Exercise 7 **Model Checking** 13 marks

1. **(4 marks)** Model check the CTL proposition $E[E[\neg p\,U\,q]\,U\,E[\neg q\,U\,p]]$ against the transition system



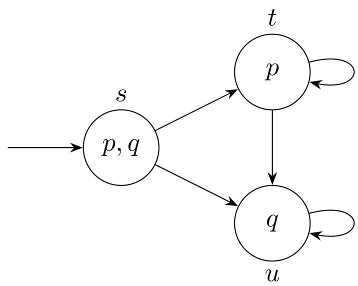
Make sure that you clearly label every node with every subformula that it satisfies.

2. **(3 marks)** Consider the transition system



In the subgraph of states that are labelled p, which are the Strongly Connected Components? Secondly, which states satisfy the CTL formula EGp?

3. **(6 marks)** Transform the LTL proposition $Gp \vee Gq$ to the simplest possible equivalent proposition that uses only the connectives $\top, \neg, \wedge, U$ (where $\top$ is shorthand for a proposition that holds at any state). Then model check this proposition against the transition system



Make sure that you clearly label every node with every subformula that it satisfies. If you erase any states or transitions as part of your working, you should draw a new diagram instead of crossing things out, so that your marker can follow your development.

Appendix

Truth Tables

	$\bot$
	0

p	$\neg p$
1	0
0	1

p	q	$p \land q$	$p \lor q$	$p \to q$
1	1	1	1	1
1	0	0	1	0
0	1	0	1	1
0	0	0	0	1

Natural Deduction

$$\frac{}{\varphi \vdash \varphi}\ A$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \land \psi}\ \land I \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi}\ \land E1 \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi}\ \land E2$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \psi}\ \to E$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \neg I \qquad \frac{\Gamma \vdash \neg\varphi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \bot}\ \neg E \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot E$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \lor \psi}\ \lor I1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \lor \psi}\ \lor I2 \qquad \frac{\Gamma \vdash \varphi \lor \psi \quad \Gamma', \varphi \vdash \sigma \quad \Gamma'', \psi \vdash \sigma}{\Gamma, \Gamma', \Gamma'' \vdash \sigma}\ \lor E$$

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ \neg\neg E$$

$$\frac{}{\vdash t = t}\ = I \qquad \frac{\Gamma \vdash t = u \quad \Gamma' \vdash \varphi[t/x]}{\Gamma, \Gamma' \vdash \varphi[u/x]}\ = E$$

$$\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x \varphi}\ \exists I \qquad a \notin FV(\Gamma, \varphi, \Gamma', \psi) : \ \frac{\Gamma \vdash \exists x \varphi \quad \Gamma', \varphi[a/x] \vdash \psi}{\Gamma, \Gamma' \vdash \psi}\ \exists E$$

$$a \notin FV(\Gamma, \varphi) : \ \frac{\Gamma \vdash \varphi[a/x]}{\Gamma \vdash \forall x \varphi}\ \forall I \qquad \frac{\Gamma \vdash \forall x \varphi}{\Gamma \vdash \varphi[t/x]}\ \forall E$$

First Order Logic: Free Variables and Substitution

- $FV(P(t_1, \ldots, t_n))$ and $FV(t = u)$ are all variables appearing in any of the terms.
- $FV(\bot)$ is $\emptyset$
- $FV(\neg\varphi)$ is $FV(\varphi)$
- $FV(\varphi \bullet \psi)$ is $FV(\varphi) \cup FV(\psi)$ for $\bullet \in \{\land, \lor, \to\}$
- $FV(\heartsuit x\, \varphi)$ is $FV(\varphi) - \{x\}$ for $\heartsuit \in \{\forall, \exists\}$

- $x[t/x]$ is t
- $y[t/x]$ is y
- $f(t_1, \ldots, t_n)[t/x]$ is $f(t_1[t/x], \ldots, t_n[t/x])$
- $P(t_1, \ldots, t_n)[t/x]$ is $P(t_1[t/x], \ldots, t_n[t/x])$
- $(u = u')[t/x]$ is $u[t/x] = u'[t/x]$
- $\bot[t/x]$ is $\bot$
- $(\neg\varphi)[t/x]$ is $\neg(\varphi[t/x])$
- $(\varphi \bullet \psi)[t/x]$ is $(\varphi[t/x]) \bullet (\psi[t/x])$ for $\bullet \in \{\land, \lor, \to\}$
- $(\heartsuit y \varphi)[t/x]$ is $\heartsuit y(\varphi[t/x])$ for $\heartsuit \in \{\forall, \exists\}$, so long as x and y are different and y is not a free variable in t

First Order Logic Semantics

- $\models_{\mathcal{M},e} P(t_1, \ldots, t_n)$ if $(t_1^{\mathcal{M},e}, \ldots, t_n^{\mathcal{M},e}) \in P^{\mathcal{M}}$
- $\models_{\mathcal{M},e} t = u$ if $t^{\mathcal{M},e} = u^{\mathcal{M},e}$ in the universe of discourse D
- $\models_{\mathcal{M},e} \bot$ never
- $\models_{\mathcal{M},e} \neg\varphi$ if it is not the case that $\models_{\mathcal{M},e} \varphi$
- $\models_{\mathcal{M},e} \varphi \wedge \psi$ if $\models_{\mathcal{M},e} \varphi$ and $\models_{\mathcal{M},e} \psi$
- $\models_{\mathcal{M},e} \varphi \vee \psi$ if $\models_{\mathcal{M},e} \varphi$ or $\models_{\mathcal{M},e} \psi$ (or both)
- $\models_{\mathcal{M},e} \varphi \to \psi$ if $\models_{\mathcal{M},e} \varphi$ implies $\models_{\mathcal{M},e} \psi$
- $\models_{\mathcal{M},e} \forall x\varphi$ if for all $d \in D$ we have $\models_{\mathcal{M},e[x\mapsto d]} \varphi$
- $\models_{\mathcal{M},e} \exists x\varphi$ if there exists $d \in D$ such that $\models_{\mathcal{M},e[x\mapsto d]} \varphi$

Logic4Fun Built-in Syntax

- `function NOT (bool): bool.`
- `function AND (bool,bool): bool.`
- `function OR (bool,bool): bool.`
- `function XOR (bool,bool): bool.`
- `function IMP (bool,bool): bool.`
- `function IFF (bool,bool): bool.`
- `function ALL (var:S) (bool): bool.`
- `function SOME (var:S) (bool): bool.`
- `predicate = (S,S).`
- `predicate < (S,S).`
- `predicate <= (S,S).`
- `predicate > (S,S).`
- `predicate >= (S,S).`
- `predicate <> (S,S).`
- `predicate EST (S).`
- `function + (S,natnum): S`
- `function - (S,natnum): S`
- `function PRED (S) : S.`
- `function SUCC (S) : S.`
- `function DIF (S,S): natnum.`
- `name MIN : S.`
- `name MAX : S.`

Tableaux

$$\frac{\mathbf{T}:\bot}{\times} \qquad \frac{\mathbf{T}:\neg\varphi}{\mathbf{F}:\varphi} \qquad \frac{\mathbf{F}:\neg\varphi}{\mathbf{T}:\varphi} \qquad \frac{\mathbf{T}:\varphi\vee\psi}{\mathbf{T}:\varphi \qquad \mathbf{T}:\psi} \qquad \frac{\mathbf{F}:\varphi\vee\psi}{\begin{array}{c}\mathbf{F}:\varphi\\ \mathbf{F}:\psi\end{array}}$$

$$\frac{\mathbf{T}:\varphi\wedge\psi}{\begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{T}:\psi\end{array}} \qquad \frac{\mathbf{F}:\varphi\wedge\psi}{\mathbf{F}:\varphi \qquad \mathbf{F}:\psi} \qquad \frac{\mathbf{T}:\varphi\to\psi}{\mathbf{F}:\varphi \qquad \mathbf{T}:\psi} \qquad \frac{\mathbf{F}:\varphi\to\psi}{\begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{F}:\psi\end{array}}$$

$$\frac{\mathbf{T}:\forall x\varphi}{\begin{array}{c}\mathbf{T}:\varphi[a_1/x]\\ \mathbf{T}:\varphi[a_2/x]\\ \vdots\\ \mathbf{T}:\varphi[a_n/x]\end{array}} \qquad \frac{\mathbf{F}:\exists x\varphi}{\begin{array}{c}\mathbf{F}:\varphi[a_1/x]\\ \mathbf{F}:\varphi[a_2/x]\\ \vdots\\ \mathbf{F}:\varphi[a_n/x]\end{array}}$$

where $a_1, \ldots, a_n$ are all variables in the tableau appearing free before or after this line. If no variables appear free before this line, the conclusion is $\varphi[a/x]$, along with $\varphi[b/x]$ for any other variable b appearing free after this line.

$$\frac{\mathbf{F}:\forall x\varphi}{\mathbf{F}:\varphi[a/x]} \qquad \frac{\mathbf{T}:\exists x\varphi}{\mathbf{T}:\varphi[a/x]}$$

where a does not appear free earlier in the tableau.

Branching quantifier rules:

$$\frac{\mathbf{F}:\forall x\varphi}{\mathbf{F}:\varphi[a_1/x] \quad \cdots \quad \mathbf{F}:\varphi[a_n/x] \quad \mathbf{F}:\varphi[a/x]} \qquad \frac{\mathbf{T}:\exists x\varphi}{\mathbf{T}:\varphi[a_1/x] \quad \cdots \quad \mathbf{T}:\varphi[a_n/x] \quad \mathbf{T}:\varphi[a/x]}$$

where $a_1, \ldots, a_n$ are all terms (= variables) in the tableau appearing free before or after this line, and a does not appear free earlier in the tableau.

$$\frac{\mathbf{T}:G\varphi}{\begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{T}:XG\varphi\end{array}} \qquad \frac{\mathbf{F}:G\varphi}{\mathbf{F}:\varphi \qquad \mathbf{F}:XG\varphi} \qquad \frac{\mathbf{T}:F\varphi}{\mathbf{T}:\varphi \qquad \mathbf{T}:XF\varphi} \qquad \frac{\mathbf{F}:F\varphi}{\begin{array}{c}\mathbf{F}:\varphi\\ \mathbf{F}:XF\varphi\end{array}}$$

$$\frac{\mathbf{T}:\varphi\,U\,\psi}{\mathbf{T}:\psi \qquad \begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{T}:X(\varphi\,U\,\psi)\end{array}} \qquad \frac{\mathbf{F}:\varphi\,U\,\psi}{\begin{array}{c}\mathbf{F}:\psi\\ \mathbf{F}:\varphi \qquad \mathbf{F}:X(\varphi\,U\,\psi)\end{array}}$$

If the node is poised, you may step:

$$\frac{\mathbf{T}:X\varphi_1 \quad \cdots \quad \mathbf{T}:X\varphi_m \quad \mathbf{F}:X\psi_1 \quad \cdots \quad \mathbf{T}:X\psi_n}{\begin{array}{c}\Downarrow\\ \mathbf{T}:\varphi_1, \cdots, \mathbf{T}:\varphi_m, \mathbf{F}:\psi_1, \cdots, \mathbf{F}:\psi_n\end{array}}$$

Semantics of Temporal Logics

LTL:

- $\vDash_{\mathcal{M},\sigma} p$ if $p \in L(\sigma_0)$
- $\vDash_{\mathcal{M},\sigma} \bot$ never
- $\vDash_{\mathcal{M},\sigma} \neg\varphi$ if it is not the case that $\vDash_{\mathcal{M},\sigma} \varphi$
- $\vDash_{\mathcal{M},\sigma} \varphi\wedge\psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ and $\vDash_{\mathcal{M},\sigma} \psi$
- $\vDash_{\mathcal{M},\sigma} \varphi\vee\psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ or $\vDash_{\mathcal{M},\sigma} \psi$ (or both)
- $\vDash_{\mathcal{M},\sigma} \varphi\to\psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ implies $\vDash_{\mathcal{M},\sigma} \psi$
- $\vDash_{\mathcal{M},\sigma} X\varphi$ if $\vDash_{\mathcal{M},\sigma_{\geq 1}} \varphi$
- $\vDash_{\mathcal{M},\sigma} G\varphi$ if for all natural numbers i, $\vDash_{\mathcal{M},\sigma_{\geq i}} \varphi$

- $\vDash_{\mathcal{M},\sigma} F\varphi$ if there exists a natural number i such that $\vDash_{\mathcal{M},\sigma_{\geq i}} \varphi$
- $\vDash_{\mathcal{M},\sigma} \varphi\, U\, \psi$ if there exists a natural number i such that $\vDash_{\mathcal{M},\sigma_{\geq i}} \psi$, and for all $h < i$ we have $\vDash_{\mathcal{M},\sigma_{\geq h}} \varphi$

CTL:

- $\vDash_{\mathcal{M},s} p$ if $p \in L(s)$
- $\vDash_{\mathcal{M},s} \bot$ never
- $\vDash_{\mathcal{M},s} \neg\varphi$ if it is not the case that $\vDash_{\mathcal{M},s} \varphi$
- $\vDash_{\mathcal{M},s} \varphi \wedge \psi$ if $\vDash_{\mathcal{M},s} \varphi$ and $\vDash_{\mathcal{M},s} \psi$
- $\vDash_{\mathcal{M},s} \varphi \vee \psi$ if $\vDash_{\mathcal{M},s} \varphi$ or $\vDash_{\mathcal{M},s} \psi$ (or both)
- $\vDash_{\mathcal{M},s} \varphi \to \psi$ if $\vDash_{\mathcal{M},s} \varphi$ implies $\vDash_{\mathcal{M},s} \psi$
- $\vDash_{\mathcal{M},s} AX\varphi$ if for all transitions $s \to s'$ we have $\vDash_{\mathcal{M},s'} \varphi$.
- $\vDash_{\mathcal{M},s} EX\varphi$ if there exists a transition $s \to s'$ such that $\vDash_{\mathcal{M},s'} \varphi$
- $\vDash_{\mathcal{M},s} AG\varphi$ if all paths $s \to \cdots$ and all states s' in such a path, $\vDash_{\mathcal{M},s'} \varphi$
- $\vDash_{\mathcal{M},s} EG\varphi$ if there exists a path $s \to \cdots$ such that for all states s' in that path, $\vDash_{\mathcal{M},s'} \varphi$
- $\vDash_{\mathcal{M},s} AF\varphi$ if all for paths $s \to \cdots$ there exists a state s' in that path such that $\vDash_{\mathcal{M},s'} \varphi$
- $\vDash_{\mathcal{M},s} EF\varphi$ if there exists a path $s \to \cdots$ such that there exists a state s' in that path such that $\vDash_{\mathcal{M},s'} \varphi$
- $\vDash_{\mathcal{M},s} A[\varphi\, U\, \psi]$ if for all paths $s \to \cdots$ there exists a state s' in that path such that $\vDash_{\mathcal{M},s'} \psi$, and for all strictly earlier states s'' in that path, $\vDash_{\mathcal{M},s''} \varphi$
- $\vDash_{\mathcal{M},s} E[\varphi\, U\, \psi]$ if there exists a path $s \to \cdots$ such that there exists a state s' in that path such that $\vDash_{\mathcal{M},s'} \psi$, and for all strictly earlier states s'' in that path, $\vDash_{\mathcal{M},s''} \varphi$

CTL* path propositions are exactly as for LTL, along with the case where φ is a CTL* proposition:

- $\vDash_{\mathcal{M},\sigma} \varphi$ if $\vDash_{\mathcal{M},\sigma_0} \varphi$

CTL* propositions are exactly as for CTL for variables and propositional connectives, along with:

- $\vDash_{\mathcal{M},s} A[\alpha]$ if for all paths $\sigma = s \to \cdots$, $\vDash_{\mathcal{M},\sigma} \alpha$
- $\vDash_{\mathcal{M},s} E[\alpha]$ if there exists a path $\sigma = s \to \cdots$ such that $\vDash_{\mathcal{M},\sigma} \alpha$

Model Checking

CTL:

- Label with propositional variables according to the labelling function
- Label nothing with $\bot$
- If a state is not labelled with φ, label it with $\neg\varphi$
- If a state is labelled with both φ and ψ, label it with $\varphi \wedge \psi$
- Label a state with $EX\varphi$ if any of its successors are labelled with φ
- Label with $E[\varphi\, U\, \psi]$ all states that are labelled with ψ, then label with $E[\varphi\, U\, \psi]$ all states that are labelled with φ for which some immediate successor is labelled with $E[\varphi\, U\, \psi]$, until done

- Label with $EG\varphi$ all states that are labelled with φ, then delete $EG\varphi$ from any state with no successors labelled with $EG\varphi$, until done, *or*
- Label with $EG\varphi$ any state in the subgraph of states labelled φ that can reach a non-trivial SCC in any number of transitions
- Label with $E_C G\varphi$ any state in the subgraph of states labelled φ that can reach a fair SCC in any number of transitions

LTL, static phase:

- No state gets label $\perp$
- A state gets label $\neg\varphi$ if and only if it does not get label φ
- A state gets label $\varphi \wedge \psi$ if and only if it gets both φ and ψ
- Make copies both with and without $X\varphi$
- Add $\varphi\, U\, \psi$, making a copy if necessary, unless a state has neither φ nor ψ.
- Leave out $\varphi\, U\, \psi$, making a copy if necessary, unless a state has ψ

LTL, transition phase: given states s', t', which are copies of s, t from the original model, add a transition $s' \to t'$ if there was a transition $s \to t$ in the original model and

- If s' has $X\varphi$ then t' must have φ
- If s' does not have $X\varphi$ then t' must not have φ
- If s' has $\varphi\, U\, \psi$ and does not have ψ then t' must have $\varphi\, U\, \psi$
- If s has φ and does not have $\varphi\, U\, \psi$ then t' must not have $\varphi\, U\, \psi$