COMP6262 (Logic) Mock Exam

Australian National University

Semester 1, 2025

Instructions

Note that there is a separate mock exam for COMP2620, although there is significant overlap with these questions.

This is a mock exam for the Logic course, created to compensate for the fact that, because of major changes to course content, prior final assessments are not relevant. The real final exam will have very similar structure and difficulty, but will cover slightly different parts of the curriculum. The real final exam will come in a format with spaces in which you should fill out answers, and also come packaged with appendices with basic rules, which will resemble the rule recaps included at the start of tutorials.

Marks add to 100, so should take on average 1.8 minutes per mark, not including the 15 minutes reading time.

1 Truth Tables [7 marks]

1. **(3 marks)** Construct the truth table for the proposition

$$p \wedge q \to \neg p$$

   Clearly indicate which column is your final answer. Is this proposition **satisfiable**? Give a brief English language explanation for your answer.

2. **(4 marks)** Construct the truth table for the sequent

$$\neg p \vee q,\ \neg(q \wedge \neg r) \vdash p \to r$$

   Clearly indicate which columns are your final answer for each proposition. Is this sequent **valid**? Give a brief English language explanation for your answer.

2 Natural Deduction [15 marks]

Prove the following sequents using natural deduction. You should use the five part notation for natural deduction: which premises are being used; a line number; a proposition or first order logic formula; previous lines used; and the rule name.

1. **(5 marks)** $\vdash \neg\neg\neg p \to \neg p$. For this question only you must only use the rules of intuitionistic logic.
2. **(5 marks)** $(p \to q) \vee \neg\neg r, p \vdash q \vee r$
3. **(5 marks)** $\forall x \forall y(Rxy \to x = y), \exists x \neg(x = c) \vdash \exists x \exists y \neg Rxy$, where c is a constant and R is a binary predicate.

3 Soundness and Completeness [14 marks]

1. **(4 marks)** Suppose that the following rule was added to our natural deduction rules:

$$\frac{\Gamma \vdash \varphi \to \psi \quad \Gamma', \psi \vdash \theta}{\Gamma, \Gamma', \varphi \vdash \theta}$$

   Use induction to show that this rule is **sound** with respect to propositional logic semantics (truth tables). Be careful to clearly state your induction hypothesis.

2. **(5 marks)** It is a fact that, if φ is a proposition without variables, which hence has a one row truth table:

   (a) if the truth table evaluation of φ is 1, then $\vdash \varphi$ can be proved with natural deduction;

   (b) if the truth table evaluation of φ is 0, then $\vdash \neg\varphi$ can be proved with natural deduction.

   Use induction to show that both of these statements are true where the main connective of φ is $\wedge$. Be careful to clearly state your induction hypothesis.

3. **(5 marks)** The substitution lemma states that, for any model $\mathcal{M}$, environment e, first order logic formula φ, term t, and variable y,

$$\vDash_{\mathcal{M},e} \varphi[t/y] \text{ if and only if } \vDash_{\mathcal{M},e[y \mapsto t^{\mathcal{M},e}]} \varphi$$

   Supposed we introduced a new quantifier to first order logic, written $\exists^\infty$, with semantics

$$\vDash_{\mathcal{M},e} \exists^\infty x\varphi \text{ if there exists infinitely many } d \text{ in the universe of discourse such that } \vDash_{\mathcal{M},e[x \mapsto d]} \varphi$$

   Show by induction that the substitution lemma holds when the main connective of φ is $\exists^\infty$. Be careful to clearly state your induction hypothesis.

4 Modelling from English [15 marks]

1. **(5 marks)** The following are the premises of a philosophical argument by Samuel Clarke (1675–1729), as summarised by George Boole in 1854. First, state which propositional variable names you will use for which concept, and second, translate each premise into propositional logic. You may use standard propositional logic notation ($\wedge, \vee$, etc.) or Logic4Fun notation (AND, OR, etc.).

   - If matter is a necessary being, either the property of gravitation is necessarily present, or it is necessarily absent.
   - If gravitation is necessarily absent, and the world is not subject to any presiding intelligence, motion does not exist.
   - If gravitation is necessarily present, a vacuum is necessary.
   - If a vacuum is necessary, matter is not a necessary being.
   - If matter is a necessary being, the world is not subject to a presiding intelligence.

   Can you detect which commonsense premise is missing that would allow you to prove, in conjunction with the other premises, that 'matter is not a necessary being'? You do not need to complete a formal proof; merely state a simple sixth premise.

2. **(6 marks)** Translate the following statements into first order logic notation or Logic4Fun notation. Be consistent in which notation you use. If using first order logic notation, be explicit about which relations or predicates you are using before you use them. If using Logic4Fun notation, be explicit about what needs to be put into the Sorts and Vocabulary boxes, as well as Constraints.

   - We are interested in creatures (including, but not limited to, ghosts) and houses. Every creature resides in exactly one house. Houses themselves do not reside in anything and nothing resides in a creature.
   - Creatures that reside alone do not scare anyone.

- No ghost would scare themselves. In fact, no ghost would even reside in the same house with a creature that does that.
- Any ghost that scares only non-ghosts will definitely scare all the non-ghosts in the house they reside in.
- Creatures that only scare creatures that do not live with them are always specialists: either scaring only ghosts, or scaring only non-ghosts.
- There have been cases of two ghosts scaring each other! This only happens when the two ghosts reside in the same house.

3. **(4 marks)** You have been hired to analyse the speeches of a politician. You start by characterising each part of their speeches as a (angry), d (detailed), h (hopeful), p (patriotic), or some combination. You intend to check each of their speeches against the rules of good political speech writing. To make this work, you must formalise these rules in LTL (no branching time logic is necessary, as you will analyse each speech as a linear succession of states corresponding to different combinations of moods, with a self loop at the conclusion).

   - Every part of a good speech should hit at least one of the four moods.
   - A speech should have a section with detailed policy proposals, but it is not good to start off too detailed.
   - If the speech gets patriotic at any stage then it should end patriotic.
   - If one is getting angry during a detailed critique of your opponents, it is always good to transition directly from that to a denunciation that is still angry, but less detailed.
   - A speech needs a hopeful part that comes after all the angry parts. Do not put any hopeful parts before then.

   You move on to employment with another politician, who likes to prepare different plans for their speeches so that they can adjust their approach depending on whether the crowd are looking excited or not (i.e., bored). Use CTL to formalise the following rules, with the variable e for the crowd being excited:

   - If the crowd have become bored there must always be a way to get them excited for the next part of the speech.
   - Angry patriotism is always exciting, so every speech plan should always keep the option open to hit those moods together.

5 Tableaux [23 marks]

For all tableaux in this section, you should number your lines; label on the right each new signed proposition or formula by which lines justify it; and cross each branch that can close, with line justifications beside any crosses.

1. **(5 marks)** Use the tableaux method to prove valid the sequent

$$(p \to q) \to p,\ q \wedge r \to \neg p \vdash \neg q \vee \neg r$$

2. **(3 marks)** Use the tableaux method to prove that the first order logic formula called the drinker's paradox is a theorem (where D is a unary predicate):

$$\exists x(Dx \to \forall y\, Dy)$$

3. **(5 marks)** Use the tableaux method to show that the following set of signed first order logic formulas is consistent (where R is a binary predicate). You do not have to explore all branches once you find one terminated open branch.

$$\mathbf{T} : \exists x \exists y Ryx,\ \mathbf{T} : \forall x \forall y (Rxy \to \neg Ryx)$$

   Extract a satisfying model.

4. **(5 marks)** Use the tableaux method to show that the following first order logic formula is not a theorem (where R is a binary predicate). You do not have to explore all branches once you find one terminated open branch.

$$\exists x(\exists y\, Rxy \rightarrow Rxx)$$

Extract a counterexample.

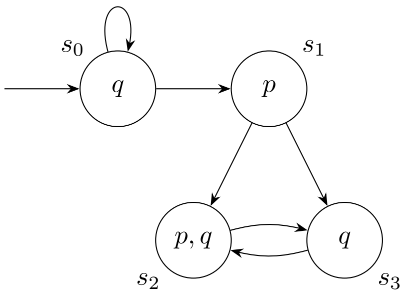
5. **(5 marks)** Use the tableaux method to prove valid the sequent

$$F(p\,U\,q) \vdash Fq$$

If your tableau becomes repetitive in parts you may leaves lines out, as long as you write clear English language explanations of what is happening.

6 Semantics [12 marks]

1. **(4 marks)** Consider the transition system



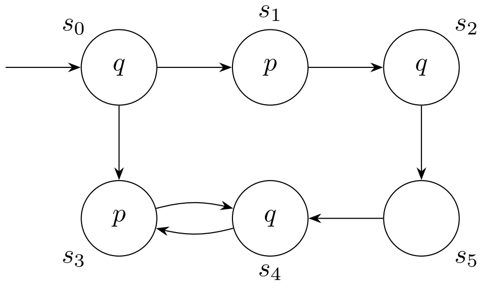
Describe a path which *does* satisfy the LTL proposition $Xp\,U\,Gq$, and then a path that *does not* satisfy that same LTL proposition. (Because paths are infinite, you will need to use words to explain what their repetitive infinite behaviour is.)

2. **(8 marks)**

- Make a mathematical argument, using the logics' semantics, that any transition system and state that satisfies the CTL* proposition $E[Fp\,U\,q]$ also satisfies the CTL proposition $E[EFp\,U\,q]$.
- Draw a transition system that satisfies $E[EFp\,U\,q]$ but does not satisfy $E[Fp\,U\,q]$. Conduct a model check to show that it does satisfy $E[EFp\,U\,q]$. Make sure that you clearly label every node with every subformula that it satisfies. Give an informal English explanation why it does not satisfy $E[Fp\,U\,q]$.

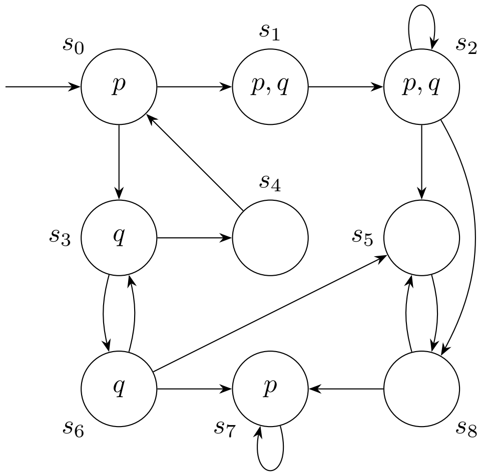
7 Model Checking [14 marks]

1. **(4 marks)** Model check the CTL proposition $EXEGE[p\,U\,q]$ against the transition system



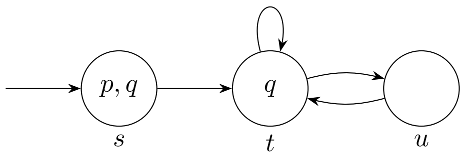
Make sure that you clearly label every node with every subformula that it satisfies.

2. **(4 marks)** Consider the transition system



Indicate in which way the states separate into Strongly Connected Components (SCCs). Indicate which SCCs are trivial. Indicate which are fair, with respect to the set of fairness constaints $C = \{p, q\}$. Finally, list all states that satisfy the proposition $E_C G\top$, where $\top$ is any proposition that holds at all states.

3. **(6 marks)** Model check the LTL proposition $(p\,U\,q)\,U\,Xq$ against the transition system



Make sure that you clearly label every node with every subformula that it satisfies. If you erase any states or transitions as part of your working, you should draw a new diagram instead of crossing things out, so that your marker can follow your development.