COMP2620/6262 (Logic) Tutorial

Week 3

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **truth tables**. The tutor will hand out blank paper, on which you should clearly write your university ID and name. The tutor will write the truth tables for connectives on the whiteboard, and then write a proposition (with three variables) on the board. You will have **ten minutes** to complete a truth table for this proposition. Please clearly indicate which column of truth values is your final answer. Other columns of calculations might be consulted to give you partial marks if you made some error in your final answer.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

Recall the natural deduction rules for propositional logic:

$$\frac{}{\varphi \vdash \varphi}\ A$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi}\ \wedge I \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge E1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E2$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \psi}\ \to E$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \neg I \qquad \frac{\Gamma \vdash \neg\varphi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \bot}\ \neg E \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot E$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee I2 \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma', \varphi \vdash \sigma \quad \Gamma'', \psi \vdash \sigma}{\Gamma, \Gamma', \Gamma'' \vdash \sigma}\ \vee E$$

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ \neg\neg E$$

1. **The test at the start of the next tutorial will resemble this question. Natural deduction rules will be provided, as above. You will be asked to prove without use of $\neg\neg E$. For full marks, you will need to use the five part notation taught in lectures: which premises are being used; a line number; a proposition; previous lines used; and the rule name.**

   Prove the following sequents by natural deduction, which do not require $\neg\neg E$. Note that there are quite a few exercises, so if you are making progress during tutorial time, skip a few and try some examples involving a different group of connectives.

- $p \wedge q, r \wedge s \;\vdash\; (p \wedge r) \wedge (q \wedge s)$

  Solution.

α_1	(1)	$p \wedge q$		A
α_2	(2)	$r \wedge s$		A
α_1	(3)	p	1	$\wedge$E1
α_1	(4)	q	1	$\wedge$E2
α_2	(5)	r	2	$\wedge$E1
α_2	(6)	s	2	$\wedge$E2
α_1, α_2	(7)	$p \wedge r$	3, 5	$\wedge$I
α_1, α_2	(8)	$q \wedge s$	4, 6	$\wedge$I
α_1, α_2	(9)	$(p \wedge r) \wedge (q \wedge s)$	7, 8	$\wedge$I

- $p \to q \wedge r \;\vdash\; (p \to q) \wedge (p \to r)$

  Solution.

α_1	(1)	$p \to q \wedge r$		A
α_2	(2)	p		A
α_1, α_2	(3)	$q \wedge r$	1, 2	$\to$E
α_1, α_2	(4)	q	3	$\wedge$E1
α_1	(5)	$p \to q$	4	$\to$I
α_1, α_2	(6)	r	3	$\wedge$E2
α_1	(7)	$p \to r$	6	$\to$I
α_1	(8)	$(p \to q) \wedge (p \to r)$	5, 7	$\wedge$I

- $(p \to q) \wedge (p \to r) \;\vdash\; p \to q \wedge r$

  Solution.

α_1	(1)	$(p \to q) \wedge (p \to r)$		A
α_2	(2)	p		A
α_1	(3)	$p \to q$	1	$\wedge$E1
α_1, α_2	(4)	q	2, 3	$\to$E
α_1	(5)	$p \to r$	1	$\wedge$E2
α_1, α_2	(6)	r	2, 5	$\to$E
α_1, α_2	(7)	$q \wedge r$	4, 6	$\wedge$I
α_1	(8)	$p \to q \wedge r$	7	$\to$I

- $p \wedge q \to r \;\vdash\; (q \to p) \to (q \to r)$

  Solution.

α_1	(1)	$(p \wedge q) \to r$		A
α_2	(2)	$q \to p$		A
α_3	(3)	q		A
α_2, α_3	(4)	p	2, 3	$\to$E
α_2, α_3	(5)	$p \wedge q$	3, 4	$\wedge$I
$\alpha_1, \alpha_2, \alpha_3$	(6)	r	1, 5	$\to$E
α_1, α_2	(7)	$q \to r$	6	$\to$I
α_1	(8)	$(q \to p) \to (q \to r)$	7	$\to$I

- $p \to (q \to r) \;\vdash\; (p \to q) \to (p \to r)$

  Solution.

α_1	(1)	$p \to (q \to r)$		A
α_2	(2)	$p \to q$		A
α_3	(3)	p		A
α_1, α_3	(4)	$q \to r$	1, 3	$\to$E
α_2, α_3	(5)	q	2, 3	$\to$E
$\alpha_1, \alpha_2, \alpha_3$	(6)	r	4, 5	$\to$E
α_1, α_2	(7)	$p \to r$	6	$\to$I
α_1	(8)	$(p \to q) \to (p \to r)$	7	$\to$I

- $p \;\vdash\; (q \to r) \to ((p \to q) \to r)$

  Solution.

α_1	(1)	p		A
α_2	(2)	$q \to r$		A
α_3	(3)	$p \to q$		A
α_1, α_3	(4)	q	$1, 3$	$\to$E
$\alpha_1, \alpha_2, \alpha_3$	(5)	r	$2, 4$	$\to$E
α_1, α_2	(6)	$(p \to q) \to r$	5	$\to$I
α_1	(7)	$(q \to r) \to ((p \to q) \to r)$	6	$\to$I

- $\neg p \;\vdash\; p \to q$

  Solution.

α_1	(1)	$\neg p$		A
α_2	(2)	p		A
α_1, α_2	(3)	$\bot$	$1, 2$	$\neg$E
α_1, α_2	(4)	q	3	$\bot$E
α_1	(5)	$p \to q$	4	$\to$I

- $p \to r,\; q \to \neg r \;\vdash\; \neg(p \wedge q)$

  Solution.

α_1	(1)	$p \to r$		A
α_2	(2)	$q \to \neg r$		A
α_3	(3)	$p \wedge q$		A
α_3	(4)	p	3	$\wedge$E1
α_1, α_3	(5)	r	$1, 4$	$\to$E
α_3	(6)	q	3	$\wedge$E2
α_2, α_3	(7)	$\neg r$	$2, 6$	$\to$E
$\alpha_1, \alpha_2, \alpha_3$	(8)	$\bot$	$5, 7$	$\neg$E
α_1, α_2	(9)	$\neg(p \wedge q)$	8	$\neg$I

- $p \vee (q \wedge r) \;\vdash\; p \vee q$

  Solution.

α_1	(1)	$p \vee (q \wedge r)$		A
α_2	(2)	p		A
α_2	(3)	$p \vee q$	2	$\vee$I1
α_4	(4)	$q \wedge r$		A
α_4	(5)	q	4	$\wedge$E1
α_4	(6)	$p \vee q$	5	$\vee$I2
α_1	(7)	$p \vee q$	$1, 3, 6$	$\vee$E

  This is our first use of disjunction in this tutorial, so we will break it down: because we have a disjunction (line 1) we separately assume each side of the disjunction (lines 2 and 4). Either way, we get to the desired conclusion (lines 3 and 6) so we use disjunction elimination (line 7) to remove our assumptions and conclude the proof.

- $p \wedge q \to r,\; p \to q \vee r \;\vdash\; p \to r$

  Solution.

α_1	(1)	$p \wedge q \to r$		A
α_2	(2)	$p \to q \vee r$		A
α_3	(3)	p		A
α_2, α_3	(4)	$q \vee r$	$2, 3$	$\to$E
α_5	(5)	q		A
α_3, α_5	(6)	$p \wedge q$	$3, 5$	$\wedge$I
$\alpha_1, \alpha_3, \alpha_5$	(7)	r	$1, 6$	$\to$E
α_8	(8)	r		A
$\alpha_1, \alpha_2, \alpha_3$	(9)	r	$4, 7, 8$	$\vee$E
α_1, α_2	(10)	$p \to r$	9	$\to$I

- $p \to \neg r,\ q \to \neg r \ \vdash \ r \to \neg(p \vee q)$

  Solution.

α_1	(1)	$p \to \neg r$		A
α_2	(2)	$q \to \neg r$		A
α_3	(3)	r		A
α_4	(4)	$p \vee q$		A
α_5	(5)	p		A
α_1, α_5	(6)	$\neg r$	$1, 5$	$\to$E
α_7	(7)	q		A
α_2, α_7	(8)	$\neg r$	$2, 7$	$\to$E
$\alpha_1, \alpha_2, \alpha_4$	(9)	$\neg r$	$4, 6, 8$	$\vee$E
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	(10)	$\bot$	$3, 9$	$\neg$E
$\alpha_1, \alpha_2, \alpha_3$	(11)	$\neg(p \vee q)$	10	$\neg$I
α_1, α_2	(12)	$r \to \neg(p \vee q)$	11	$\to$I

2. Complete the following natural deductions. These proofs will require $\neg\neg E$.

- $\vdash \ (p \to q) \vee p$

  Solution.

α_1	(1)	$\neg((p \to q) \vee p)$		A
α_2	(2)	p		A
α_2	(3)	$(p \to q) \vee p$	2	$\vee$I2
α_1, α_2	(4)	$\bot$	$1, 3$	$\neg$E
α_1, α_2	(5)	q	4	$\bot$E
α_1	(6)	$p \to q$	5	$\to$I
α_1	(7)	$(p \to q) \vee p$	6	$\vee$I1
α_1	(8)	$\bot$	$1, 7$	$\neg$E
	(9)	$\neg\neg((p \to q) \vee p)$	8	$\neg$I
	(10)	$(p \to q) \vee p$	9	$\neg\neg$E

  Note how similar this proof is to that of the Law of Excluded Middle in lectures.

- $(p \to q) \to p \ \vdash \ p$

  Solution.

α_1	(1)	$(p \to q) \to p$		A
α_2	(2)	$\neg p$		A
α_3	(3)	p		A
α_2, α_3	(4)	$\bot$	$2, 3$	$\neg$E
α_2, α_3	(5)	q	4	$\bot$E
α_2	(6)	$p \to q$	5	$\to$I
α_1, α_2	(7)	p	$1, 6$	$\to$E
α_1, α_2	(8)	$\bot$	$2, 7$	$\neg$E
α_1	(9)	$\neg\neg p$	8	$\neg$E
α_1	(10)	p	9	$\neg\neg$E

  In this proof we assume $\neg p$ on line 2 in search of a contradiction, and then assume line 3 to try to get the proposition $p \to q$ that we need to trigger our main premise, which we manage to do on lines 6 and 7. It is reasonable to get a bit stuck here because one might have left out line 2, assumed p first, then tried to show that q follows by contradiction. A good plan, but it does not work!

- $\neg p \to q \vee r \ \vdash \ \neg q \to p \vee r$ (Tricky)

  Solution.

α_1	(1)	$\neg p \rightarrow q \vee r$		A
α_2	(2)	$\neg q$		A
α_3	(3)	$\neg(p \vee r)$		A
α_4	(4)	p		A
α_4	(5)	$p \vee r$	4	$\vee$I1
α_3, α_4	(6)	$\bot$	$3, 5$	$\neg$E
α_3	(7)	$\neg p$	6	$\neg$I
α_1, α_3	(8)	$q \vee r$	$1, 7$	$\rightarrow$E
α_9	(9)	q		A
α_2, α_9	(10)	$\bot$	$2, 9$	$\neg$E
α_{11}	(11)	r		A
α_{11}	(12)	$p \vee r$	11	$\vee$I2
α_3, α_{11}	(13)	$\bot$	$3, 12$	$\neg$E
$\alpha_1, \alpha_2, \alpha_3$	(14)	$\bot$	$8, 10, 13$	$\vee$E
α_1, α_2	(15)	$\neg\neg(p \vee r)$	14	$\neg$I
α_1, α_2	(16)	$p \vee r$	15	$\neg\neg$E
α_1	(17)	$\neg q \rightarrow p \vee r$	16	$\rightarrow$I

Here we assume line 3 in search of a contradiction. Lines 3 to 7 shows that $\neg(p \vee r)$ gives us $\neg p$, which we knew to look for because it is the first part of the sequent's premise. We get the disjunction $q \vee r$ from this, and then argue by cases (lines 8 to 14) that we have a contradiction either way. Hence we can reject the assumption on line 3 and complete our proof.