COMP2620/6262 (Logic) Tutorial

Week 4

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **natural deduction** for propositional logic without double negation elimination. Your tutor will hand out blank paper, on which you should clearly write your university ID and name. Your tutor will also hand out paper with all natural deduction rules except $\neg\neg E$. They will then write a sequent on the board which you should attempt to prove with these rules. You should use the five part notation for natural deduction: which premises are being used; a line number; a proposition; previous lines used; and the rule name. You will have **twelve minutes** to attempt this proof.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

This tutorial will involve both the truth tables and the natural deduction rules for propositional logic:

p	q	$\bot$	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
1	1	0	0	1	1	1
1	0			0	1	0
0	1		1	0	1	1
0	0			0	0	1

$$\frac{}{\varphi \vdash \varphi}\ A$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi}\ \wedge I \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge E1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E2$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \psi}\ \to E$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \neg I \qquad \frac{\Gamma \vdash \neg\varphi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \bot}\ \neg E \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot E$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee I2 \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma', \varphi \vdash \sigma \quad \Gamma'', \psi \vdash \sigma}{\Gamma, \Gamma', \Gamma'' \vdash \sigma}\ \vee E$$

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ \neg\neg E$$

1. The soundness of natural deduction with respect to truth tables is proved by induction on the length of proofs. We assume that anything proved by a shorter natural deduction proof is valid by the method of truth tables, and then show for each natural deduction rule separately that the conclusion of that rule is also a sequent that is valid by truth table. A number of cases were proved in lectures.

   Prove the soundness of the following natural deduction rules:

- $\wedge I$

  Solution. Assume for induction that for any truth table row for which all propositions of Γ are 1, φ is 1. Assume for induction the same thing for Γ' and ψ. We are interested in truth table rows for which all of Γ and Γ' are 1. In all such rows both φ and ψ are 1. So by the truth table for $\wedge$, $\varphi \wedge \psi$ is 1.

- $\neg\neg E$

  Solution. Assume that for any truth table row for which all propositions of Γ are 1, $\neg\neg\varphi$ is 1. By the truth table for $\neg$, $\neg\varphi$ is 0, and similarly φ must be 1.

- $\neg I$

  Solution. Assume that for any truth table row for which all propositions of Γ are 1, and also φ is 1, $\bot$ is 1. But $\bot$ is never 1, so it must be for every row of the truth table either that something in Γ is 0, or that φ is 0. If something in Γ is 0 we are done with that row because we are only interested in rows for which all of Γ is 1. On the other hand if φ is 0, then $\neg\varphi$ is 1.

- $\vee E$

  Solution. Assume that for any truth table row for which all propositions of Γ are 1, $\varphi \vee \psi$ is 1. Assume also that where all propositions of Γ' are 1, and also φ is 1, σ is 1. Make the same assumption for Γ'' and ψ. We are interested in rows for which all of Γ, Γ', and Γ'' are 1. In such a row $\varphi \vee \psi$ is 1. By the truth table for $\vee$, either φ or ψ (or both) are 1. Without loss of generality say that φ is 1. Then σ is 1 on that row by our second assumption (if it were ψ that were 1, we use the third assumption).

2. The 'main lemma' in the lecture proof of completeness states that, given any proposition φ,

   (a) if φ is 1 in the k'th row of its truth table, then $\pi_k \vdash \varphi$ can be proved by natural deduction;

   (b) If φ is 0 in the k'th row of its truth table, then $\pi_k \vdash \neg\varphi$ can be proved by natural deduction.

   where π_k is a big conjunction of all propositions, or their negation, according the 1s and 0s of the k'th row of the truth table.

   This lemma is proved by induction on the length of the proposition. In lectures we proved the cases of variables, $\bot$, $\neg$, and $\vee$.

   Prove the step case, for the (a) case only, of this lemma for a proposition of form $\varphi \to \psi$. Hint: consider the different reasons why $\varphi \to \psi$ might be 1 in the k'th row of the truth table.

   Solution. Suppose for induction that the lemma (both parts) holds for φ and for ψ. Suppose that $\varphi \to \psi$ is 1 in the k'th row of the truth table. There are three possibilities to consider: φ is 1 and ψ is 1; φ is 0 and ψ is 1; and φ is 0 and ψ is 0.

   First and second case: If ψ is 1 then by induction $\pi_k \vdash \psi$ has a proof. We are always allowed to assume more premises than we need (weakening), so $\pi_k, \varphi \vdash \psi$ has a proof. But then by $\to I$ we prove $\pi_k \vdash \varphi \to \psi$.

   Third and (redundantly) second case: if φ is 0 then by induction $\pi_k \vdash \neg\varphi$ has a proof. By Assumption $\varphi \vdash \varphi$. So by $\neg E$, $\pi_k, \varphi \vdash \bot$. Then by $\bot E$, $\pi_k, \varphi \vdash \psi$. We finish with $\to I$ again.

3. (A bit time consuming, so you should attempt some of the Logic4Fun questions below first) Recall the Sheffer stroke from the week 2 tutorial:

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

   We do not need the Sheffer stroke in our basic calculus because we can encode $p \uparrow q$ as $\neg(p \wedge q)$. But if we did wish to have it as a connective, we suggest the following natural deduction rules[1]:

[1] From Richard Zach. "Natural deduction for the Sheffer stroke and Peirce's arrow (and any other truth-functional connective)." Journal of Philosophical Logic 45.2 (2016): 183-197.

$$\frac{\Gamma, \varphi, \psi \vdash \bot}{\Gamma \vdash \varphi \uparrow \psi} \uparrow I \qquad \frac{\Gamma \vdash \varphi \uparrow \psi \quad \Gamma' \vdash \varphi \quad \Gamma'' \vdash \psi}{\Gamma, \Gamma', \Gamma'' \vdash \bot} \uparrow E$$

Extend the proofs of soundness and completeness to incorporate the Sheffer stroke.

Solution.

Soundness of $\uparrow I$: Suppose for each truth table row, either something in Γ, or φ, or ψ, is 0 (because it cannot be that $\bot$ is 1). We are only interested in rows that satisfy all of Γ, so we can ignore rows in which something in Γ is 0. But if either φ or ψ, is 0, then $\varphi \uparrow \psi$ is 1.

Soundness of $\uparrow E$: Suppose on all rows for which all of Γ is 1, $\varphi \uparrow \psi$ is 1, and similarly for Γ' and φ, and Γ'' and ψ. Then on any row satisfying all of $\Gamma, \Gamma', \Gamma''$ we have all three of these propositions as 1, but by the truth table this is impossible, so there is no row satisfying all these premises, so the conclusion sequent is vacuously semantically valid.

(a) case of main lemma: Suppose for induction that the lemma holds for φ and ψ. Suppose on the k'th truth table row, $\varphi \uparrow \psi$ is 1. Then either φ or ψ is 0. Without loss of generality, say φ is. By induction we can prove $\pi_k \vdash \neg\varphi$. By Assumption, $\varphi \vdash \varphi$, so by $\neg E$, $\pi_k, \varphi \vdash \bot$. This weakens to $\pi_k, \varphi, \psi \vdash \bot$, so we use $\uparrow I$ to get $\pi_k \vdash \varphi \uparrow \psi$.

(b) case of main lemma: Suppose on the k'th truth table row, $\varphi \uparrow \psi$ is 0. Then both φ and ψ are 1, so by induction we can prove $\pi_k \vdash \varphi$ and $\pi_k \vdash \psi$. Assume for contradiction $\varphi \uparrow \psi$. Then by $\uparrow E$ we have $\pi_k, \varphi \uparrow \psi \vdash \bot$. Then by $\neg I$ we have $\pi_k \vdash \neg(\varphi \uparrow \psi)$.

4. The remainder of this tutorial involves the **Logic4Fun** website. Follow the instructions in Wattle to log on to this website and join the current class, if you have not done so already.

   Navigate to the 'Solver' and try writing some propositions of propositional logic in the Constraints box, and clicking 'Solve' to test them. You can leave the 'Sorts' and 'Vocabulary' boxes empty for now, although you might get some warnings about this.

   You can access the usual logical symbols by clicking the buttons, but you might find it easier to write `NOT`, `OR`, `AND`, and `IMP`. Because there is no $\bot$ symbol, you have to write `FALSE` for this. You must end each line with a full stop.

5. Navigate to the 'Puzzles' section and attempt the first five puzzles:

   - Integer Equalities

     Solution.

     Vocabulary:
     ```
     name {
       mine : natnum.
       yours : natnum.
     }
     ```
     Constraints:
     ```
     mine + yours = 10.
     mine - yours = 6.
     ```
     We can leave the Sorts box empty here if we do not mind receiving a few warnings, as Logic4Fun can infer that the names are intended to be natural numbers. But it is worth your while learning how to get rid of these warnings, so that you can help Logic4Fun out when it is not able to infer your sorts.

   - Integer Inequalities

     Solution.

     Vocabulary:
     ```
     name {
       Able : natnum.
       Baker : natnum.
       Charlie : natnum.
       Dog : natnum.
     }
     ```
     Constraints:

```
Able > 0.
Baker < 7.
Charlie > 3.
Dog > Able AND Dog < 4.
Charlie + Dog > Baker.
Able + Charlie < Baker.
```

- Absolute Difference

  Solution.

  Vocabulary:

  ```
  name {
    Tom : natnum.
    Dick : natnum.
    Harry : natnum.
  }
  ```

  Constraints:

  ```
  Tom + Dick + Harry = 25.
  Tom DIF Dick = (Dick DIF Harry) - 1.
  (Tom DIF Harry) DIF Dick = 1.
  Tom > (Dick DIF Harry).
  Dick > 6.
  ```

- Four Trees

  Solution.

  In Constraints, under the comment line:

  ```
  (position oak) DIF (position ash) = 2.
  position fir < position elm AND position fir < position oak.
  NOT (position ash = 1 OR position ash = 4).
  ```

- Jellybeans

  Solution.

  Sorts:

  ```
  child enum : Alice, Boris, Claire, David.
  ```

  Vocabulary:

  ```
  function {
    age (child) : natnum all_different.
    beans (child) : natnum all_different.
  }
  ```

  Constraints:

  ```
  ALL x (age x > 5 AND age x < 10).
  ALL x (beans x > 0 AND beans x < 5).
  age Alice + beans Alice = age Boris + beans Boris.
  age Claire = beans Alice + 2.
  age Alice = beans Alice + beans Claire.
  ```

The test at the start of the next tutorial will resemble these questions, most closely resembling the Absolute Difference puzzle, but with more use of propositional logic (not first order) connectives as practiced in this tutorial's question 4.