COMP2620/6262 (Logic) Tutorial

Week 4

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **natural deduction** for propositional logic without double negation elimination. Your tutor will hand out blank paper, on which you should clearly write your university ID and name. Your tutor will also hand out paper with all natural deduction rules except $\neg\neg E$. They will then write a sequent on the board which you should attempt to prove with these rules. You should use the five part notation for natural deduction: which premises are being used; a line number; a proposition; previous lines used; and the rule name. You will have **twelve minutes** to attempt this proof.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

This tutorial will involve both the truth tables and the natural deduction rules for propositional logic:

p	q	$\bot$	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
1	1	0	0	1	1	1
1	0			0	1	0
0	1		1	0	1	1
0	0			0	0	1

$$\frac{}{\varphi \vdash \varphi}\ A$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi}\ \wedge I \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge E1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge E2$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \psi}\ \to E$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \neg I \qquad \frac{\Gamma \vdash \neg\varphi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \bot}\ \neg E \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot E$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee I1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee I2 \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma', \varphi \vdash \sigma \quad \Gamma'', \psi \vdash \sigma}{\Gamma, \Gamma', \Gamma'' \vdash \sigma}\ \vee E$$

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\ \neg\neg E$$

1. The soundness of natural deduction with respect to truth tables is proved by induction on the length of proofs. We assume that anything proved by a shorter natural deduction proof is valid by the method of truth tables, and then show for each natural deduction rule separately that the conclusion of that rule is also a sequent that is valid by truth table. A number of cases were proved in lectures.

   Prove the soundness of the following natural deduction rules:

- $\wedge I$
- $\neg\neg E$
- $\neg I$
- $\vee E$

2. The 'main lemma' in the lecture proof of completeness states that, given any proposition φ,

   (a) if φ is 1 in the k'th row of its truth table, then $\pi_k \vdash \varphi$ can be proved by natural deduction;

   (b) If φ is 0 in the k'th row of its truth table, then $\pi_k \vdash \neg\varphi$ can be proved by natural deduction.

   where π_k is a big conjunction of all propositions, or their negation, according the 1s and 0s of the k'th row of the truth table.

   This lemma is proved by induction on the length of the proposition. In lectures we proved the cases of variables, $\bot$, $\neg$, and $\vee$.

   Prove the step case, for the (a) case only, of this lemma for a proposition of form $\varphi \to \psi$. Hint: consider the different reasons why $\varphi \to \psi$ might be 1 in the k'th row of the truth table.

3. (A bit time consuming, so you should attempt some of the Logic4Fun questions below first) Recall the Sheffer stroke from the week 2 tutorial:

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

   We do not need the Sheffer stroke in our basic calculus because we can encode $p \uparrow q$ as $\neg(p \wedge q)$. But if we did wish to have it as a connective, we suggest the following natural deduction rules[1]:

$$\frac{\Gamma, \varphi, \psi \vdash \bot}{\Gamma \vdash \varphi \uparrow \psi} \uparrow I \qquad \frac{\Gamma \vdash \varphi \uparrow \psi \quad \Gamma' \vdash \varphi \quad \Gamma'' \vdash \psi}{\Gamma, \Gamma', \Gamma'' \vdash \bot} \uparrow E$$

   Extend the proofs of soundness and completeness to incorporate the Sheffer stroke.

4. The remainder of this tutorial involves the **Logic4Fun** website. Follow the instructions in Wattle to log on to this website and join the current class, if you have not done so already.

   Navigate to the 'Solver' and try writing some propositions of propositional logic in the Constraints box, and clicking 'Solve' to test them. You can leave the 'Sorts' and 'Vocabulary' boxes empty for now, although you might get some warnings about this.

   You can access the usual logical symbols by clicking the buttons, but you might find it easier to write `NOT`, `OR`, `AND`, and `IMP`. Because there is no $\bot$ symbol, you have to write `FALSE` for this. You must end each line with a full stop.

5. Navigate to the 'Puzzles' section and attempt the first five puzzles:

   - Integer Equalities
   - Integer Inequalities
   - Absolute Difference
   - Four Trees
   - Jellybeans

   The test at the start of the next tutorial will resemble these questions, most closely resembling the Absolute Difference puzzle, but with more use of propositional logic (not first order) connectives as practiced in this tutorial's question 4.

[1] From Richard Zach. "Natural deduction for the Sheffer stroke and Peirce's arrow (and any other truth-functional connective)." Journal of Philosophical Logic 45.2 (2016): 183-197.