COMP2620/6262 (Logic) Tutorial

Week 6

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **natural deduction** for first order logic. Your tutor will hand out blank paper, on which you should clearly write your university ID and name. Your tutor will also hand out paper with all natural deduction rules They will then write a sequent on the board which you should attempt to prove with these rules. You should use the five part notation for natural deduction: which premises are being used; a line number; a formula; previous lines used; and the rule name. You will have **eighteen minutes** to attempt this proof.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

This tutorial involves the natural deduction rules and the semantics for first order logic:

$$\frac{}{\varphi \vdash \varphi}\, A$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi} \wedge I \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \wedge E1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \wedge E2$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to I \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \psi} \to E$$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi} \neg I \qquad \frac{\Gamma \vdash \neg\varphi \quad \Gamma' \vdash \varphi}{\Gamma, \Gamma' \vdash \bot} \neg E \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \bot E$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \vee I1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \vee I2 \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma', \varphi \vdash \sigma \quad \Gamma'', \psi \vdash \sigma}{\Gamma, \Gamma', \Gamma'' \vdash \sigma} \vee E$$

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \neg\neg E$$

$$\frac{}{\vdash t = t} = I \qquad \frac{\Gamma \vdash t = u \quad \Gamma' \vdash \varphi[t/x]}{\Gamma, \Gamma' \vdash \varphi[u/x]} = E$$

$$\frac{\Gamma \vdash \varphi[t/x]}{\Gamma \vdash \exists x \varphi} \exists I \qquad a \notin FV(\Gamma, \varphi, \Gamma', \psi) : \frac{\Gamma \vdash \exists x \varphi \quad \Gamma', \varphi[a/x] \vdash \psi}{\Gamma, \Gamma' \vdash \psi} \exists E$$

$$a \notin FV(\Gamma, \varphi) : \frac{\Gamma \vdash \varphi[a/x]}{\Gamma \vdash \forall x \varphi} \forall I \qquad \frac{\Gamma \vdash \forall x \varphi}{\Gamma \vdash \varphi[t/x]} \forall E$$

We define $\vDash_{\mathcal{M},e} \varphi$, saying that φ is *satisfied* by $(\mathcal{M}, e)$, as:

- $\vDash_{\mathcal{M},e} P(t_1, \ldots, t_n)$ if $(t_1^{\mathcal{M},e}, \ldots, t_n^{\mathcal{M},e}) \in P^{\mathcal{M}}$
- $\vDash_{\mathcal{M},e} t = u$ if $t^{\mathcal{M},e} = u^{\mathcal{M},e}$ in the universe of discourse D

- $\vDash_{\mathcal{M},e} \bot$ never
- $\vDash_{\mathcal{M},e} \neg\varphi$ if it is not the case that $\vDash_{\mathcal{M},e} \varphi$
- $\vDash_{\mathcal{M},e} \varphi \wedge \psi$ if $\vDash_{\mathcal{M},e} \varphi$ and $\vDash_{\mathcal{M},e} \psi$
- $\vDash_{\mathcal{M},e} \varphi \vee \psi$ if $\vDash_{\mathcal{M},e} \varphi$ or $\vDash_{\mathcal{M},e} \psi$ (or both)
- $\vDash_{\mathcal{M},e} \varphi \to \psi$ if $\vDash_{\mathcal{M},e} \varphi$ implies $\vDash_{\mathcal{M},e} \psi$
- $\vDash_{\mathcal{M},e} \forall x\varphi$ if for all $d \in D$ we have $\vDash_{\mathcal{M},e[x\mapsto d]} \varphi$
- $\vDash_{\mathcal{M},e} \exists x\varphi$ if there exists $d \in D$ such that $\vDash_{\mathcal{M},e[x\mapsto d]} \varphi$

1. Let R be a binary predicate and f be a unary function. For each of the formulas below, ***(a)*** suggest a model and environment that satisfies it, and ***(b)*** suggest a model and environment that does not satisfy it.
   - $\forall x\, R(x, f\, x)$
   - $R(x, x) \to \bot$
   - $R(x, y) \wedge R(y, z) \to R(x, z)$
   - $\exists x\, f\, x = y$
2. The key lemma towards the proof of soundness is the substitution lemma: $\vDash_{\mathcal{M},e} \varphi[t/x]$ if and only if $\vDash_{e[x\mapsto t^{\mathcal{M},e}]} \varphi$. It is proved by induction on the structure of φ. A few cases were given lectures. Prove it for the following cases:
   - $\neg\varphi$
   - $u = v$ for terms u and v. This will require the substitution lemma for terms, $(u[t/x])^{\mathcal{M},e} = u^{\mathcal{M},e[x\mapsto t^{\mathcal{M},e}]}$.
   - $\forall y\varphi$
3. The soundness of natural deduction for first order logic is proved by induction on the length of proofs. We assume that anything proved by a shorter natural deduction proof is semantically valid (satisfied by all models and environments), and then show for each natural deduction rule separately that the conclusion of that rule is also a sequent that is semantically valid. A number of cases were proved in lectures.

   Prove the soundness of the following natural deduction rules:
   - $\wedge E1$
   - $\to E$
   - $\to I$
   - $\exists I$
   - $\exists E$ (Tricky)
4. **The test at the start of the next tutorial will resemble these questions.**

   Attempt the following puzzles from the Logic4Fun website.
   - Pet Show
   - Parking Bay
   - Transfer List
   - Trading Post (seriously time consuming, so you might like to return to it in the teaching break)