COMP2620/6262 (Logic) Tutorial

Week 8

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **tableaux** for propositional logic. Your tutor will hand out blank paper, on which you should clearly write your university ID and name. Your tutor will also hand out paper with all tableaux rules for propositional logic. They will then write a set of signed propositions on the whiteboard. You should construct a *complete* tableaux for this set of signed propositions, continuing every branch until it closes or terminates open. You should remember to number your lines; to label on the right each new signed proposition by which lines justify it; and to cross each branch that can close, with line justifications beside any crosses. You do not need to explicitly extract a satisfying model. You will have **fourteen minutes** to construct this tableau.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

This tutorial involves the tableaux rules, with branching rules for quantifiers, for first order logic:

$$\frac{\mathbf{T}:\bot}{\times} \qquad \frac{\mathbf{T}:\neg\varphi}{\mathbf{F}:\varphi} \qquad \frac{\mathbf{F}:\neg\varphi}{\mathbf{T}:\varphi} \qquad \frac{\mathbf{T}:\varphi\vee\psi}{\mathbf{T}:\varphi \qquad \mathbf{T}:\psi} \qquad \frac{\mathbf{F}:\varphi\vee\psi}{\begin{array}{c}\mathbf{F}:\varphi\\ \mathbf{F}:\psi\end{array}}$$

$$\frac{\mathbf{T}:\varphi\wedge\psi}{\begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{T}:\psi\end{array}} \qquad \frac{\mathbf{F}:\varphi\wedge\psi}{\mathbf{F}:\varphi \qquad \mathbf{F}:\psi} \qquad \frac{\mathbf{T}:\varphi\to\psi}{\mathbf{F}:\varphi \qquad \mathbf{T}:\psi} \qquad \frac{\mathbf{F}:\varphi\to\psi}{\begin{array}{c}\mathbf{T}:\varphi\\ \mathbf{F}:\psi\end{array}}$$

$$\frac{\mathbf{T}:\forall x\varphi}{\begin{array}{c}\mathbf{T}:\varphi[a_1/x]\\ \mathbf{T}:\varphi[a_2/x]\\ \vdots\\ \mathbf{T}:\varphi[a_n/x]\end{array}} \qquad \frac{\mathbf{F}:\exists x\varphi}{\begin{array}{c}\mathbf{F}:\varphi[a_1/x]\\ \mathbf{F}:\varphi[a_2/x]\\ \vdots\\ \mathbf{F}:\varphi[a_n/x]\end{array}}$$

where $a_1, \ldots, a_n$ are all terms (= variables) in the tableau appearing free before or after this line. If no variables appear free before this line, the conclusion is $\varphi[a/x]$ only.

$$\frac{\mathbf{F}:\forall x\varphi}{\mathbf{F}:\varphi[a_1/x] \quad \cdots \quad \mathbf{F}:\varphi[a_n/x] \quad \mathbf{F}:\varphi[a/x]} \qquad \frac{\mathbf{T}:\exists x\varphi}{\mathbf{T}:\varphi[a_1/x] \quad \cdots \quad \mathbf{T}:\varphi[a_n/x] \quad \mathbf{T}:\varphi[a/x]}$$

where $a_1, \ldots, a_n$ are all terms (= variables) in the tableau appearing free before or after this line, and a does not appear free earlier in the tableau.

1. **The test at the start of the next tutorial will resemble this question.**

   In the last tutorial we only briefly practiced the branching quantifier rules, so here we will do some more.

For each of these signed propositions, use the tableaux method to extract a finite satisfying model. You should not attempt to construct the whole tableaux, because you cannot, as some branches would be infinite. It suffices to find one open terminated branch. You should likewise try to avoid pursuing any branches that you think will close, or multiple open branches; this question is about being thoughtful and pursuing one path through the tableau to get to a model.

Because the **T** rule for ∀ and **F** rule for ∃ can fire more than once, your tableaux, if worked out in detail, might become quite repetitive. You may skip some repetitive steps if you justify your rules by explaining which line you got started with, which line this part of your tableau resembles, and which substitutions for bound variables you used to get there, e.g. 'from (3), as for (6), with $[b/x]$ and $[c/y]$'. You will be able to do this in the tutorial test next week also. Do not skip any steps the first time you apply one of these rules.

- $\mathbf{T} : \forall x(\exists y\, Rxy \wedge \exists y \neg Rxy)$
- $\mathbf{T} : \forall x(\exists y(Rxx \to \neg Rxy) \wedge \forall y\, Ryy)$
- $\mathbf{F} : \exists x \forall y(\neg \forall z\, Rzz \vee (Rxy \to Ryx))$

2. Turn the following descriptions of transition systems into diagrams. The states s_0 are start states.

   - $S = \{s_0, s_1, s_2, s_3\}$; $\to = \{(s_0, s_1), (s_0, s_2), (s_1, s_1), (s_1, s_3), (s_2, s_1), (s_2, s_3), (s_3, s_2)\}$; $L(s_0) = \emptyset$; $L(s_1) = \{p, q\}$; $L(s_2) = L(s_3) = \{p\}$.
   - $S = \{s_0, s_1, s_2, s_3, s_4\}$; $\to = \{(s_0, s_1), (s_0, s_2), (s_1, s_3), (s_1, s_4), (s_2, s_3), (s_2, s_4), (s_3, s_0), (s_4, s_4)\}$; $L(s_0) = L(s_1) = L(s_2) = \{p\}$; $L(s_3) = \{q\}$; $L(s_4) = \{r\}$.

3. For each of the transition systems of the previous question, suggest some paths, starting at the state states. Then for each path you suggest, suggest some LTL propositions that would be satisfied by that path. Try to use all the new connectives of LTL.

4. Recall the semantics of LTL:

   - $\vDash_{\mathcal{M},\sigma} p$ if $p \in L(\sigma_0)$
   - $\vDash_{\mathcal{M},\sigma} \bot$ never
   - $\vDash_{\mathcal{M},\sigma} \neg\varphi$ if it is not the case that $\vDash_{\mathcal{M},\sigma} \varphi$
   - $\vDash_{\mathcal{M},\sigma} \varphi \wedge \psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ and $\vDash_{\mathcal{M},\sigma} \psi$
   - $\vDash_{\mathcal{M},\sigma} \varphi \vee \psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ or $\vDash_{\mathcal{M},\sigma} \psi$ (or both)
   - $\vDash_{\mathcal{M},\sigma} \varphi \to \psi$ if $\vDash_{\mathcal{M},\sigma} \varphi$ implies $\vDash_{\mathcal{M},\sigma} \psi$
   - $\vDash_{\mathcal{M},\sigma} X\varphi$ if $\vDash_{\mathcal{M},\sigma_{\geq 1}} \varphi$
   - $\vDash_{\mathcal{M},\sigma} G\varphi$ if for all natural numbers i, $\vDash_{\mathcal{M},\sigma_{\geq i}} \varphi$
   - $\vDash_{\mathcal{M},\sigma} F\varphi$ if there exists a natural number i such that $\vDash_{\mathcal{M},\sigma_{\geq i}} \varphi$
   - $\vDash_{\mathcal{M},\sigma} \varphi\, U\, \psi$ if there exists a natural number i such that $\vDash_{\mathcal{M},\sigma_{\geq i}} \psi$, and for all $h < i$ we have $\vDash_{\mathcal{M},\sigma_{\geq h}} \varphi$

   Argue using the semantics that the following propositions are equivalent (regardless of the system $\mathcal{M}$ or path σ).

   - $X(\varphi \vee \psi)$ and $X\varphi \vee X\psi$.
   - $XG\varphi$ and $GX\varphi$.
   - $\neg\varphi\, U\, \varphi$ and $F\varphi$.
   - $\bot\, U\, \varphi$ and φ

5. Give LTL propositions to specify the desirable properties of an elevator (lift) in a two story building. In each state the elevator should be on the ground or first floor, and buttons might have been pressed, or not, by people wanting the elevator. Because there are only two floors, only one button is required on each floor and no buttons are required inside the elevator. To simplify the system you may assume that different buttons are never pressed at the exact same moment, although they might be pressed close enough together that the elevator has not been able to fulfil the first request.

   Then draw a transition system whose paths satisfy your propositions.

6. (Time consuming and open ended; no complete solution will be presented). Suggest some LTL propositions that would help to specify a three floor elevator. In particular, how do you ensure that the elevator does not keep itself busy moving up and down between two of the floors while ignoring requests for the third?

   You will find a transition system for such an elevator too large to construct in reasonable time, but you are welcome to think about how it might be done.