COMP2620/6262 (Logic) Tutorial

Week 9

Semester 1, 2025

Tutorial Quiz

In each tutorial, apart from week 2, there is a short quiz on skills practised in the previous tutorial. Your top 7 quiz attempts, out of the 9 available, will collectively count for 50% of your final mark.

This week's quiz is on **tableaux** for first order logic, with branching quantifier rules. Your tutor will hand out blank paper, on which you should clearly write your university ID and name. Your tutor will also hand out paper with all tableaux rules for first order logic, with branching quantifier rules. They will then write a signed proposition on the whiteboard. You should then use the tableaux method to extract a finite satisfying model. Be explicit about what your model (universe of discourse and interpretation of quantifiers) is. You will have **twenty minutes** to do this.

You should not attempt to construct a completed tableaux, because some branches will be infinite. You do not need to pursue branches that you think will close, or multiple open branches. If your tableau becomes repetitive due to multiple applications of certain quantifier rules, you may skip some repetitive steps so long as you justify your lines by explaining which line you got started with, which line this part of your tableau resembles, and which substitutions for bound variables you used to get there, e.g. 'from (3), as for (6), with $[b/x]$ and $[c/y]$'. Do not skip any steps the first time you apply these rules.

You are not permitted to have any other resource on the table during this quiz, including any electronic device. If you finish your quiz before time elapses you may put your hand up and your tutor will collect your sheet. Once you have done this, you may get a device out and start work silently on this week's questions. If you are still working when time elapses you must stop writing immediately and let your tutor collect your paper.

This Week's Exercises

This tutorial involves the tableaux rules for linear temporal logic:

$$\frac{\mathbf{T} : \bot}{\times} \qquad \frac{\mathbf{T} : \neg\varphi}{\mathbf{F} : \varphi} \qquad \frac{\mathbf{F} : \neg\varphi}{\mathbf{T} : \varphi} \qquad \frac{\mathbf{T} : \varphi \vee \psi}{\mathbf{T} : \varphi \quad\quad \mathbf{T} : \psi} \qquad \frac{\mathbf{F} : \varphi \vee \psi}{\begin{array}{c}\mathbf{F} : \varphi \\ \mathbf{F} : \psi\end{array}}$$

$$\frac{\mathbf{T} : \varphi \wedge \psi}{\begin{array}{c}\mathbf{T} : \varphi \\ \mathbf{T} : \psi\end{array}} \qquad \frac{\mathbf{F} : \varphi \wedge \psi}{\mathbf{F} : \varphi \quad\quad \mathbf{F} : \psi} \qquad \frac{\mathbf{T} : \varphi \to \psi}{\mathbf{F} : \varphi \quad\quad \mathbf{T} : \psi} \qquad \frac{\mathbf{F} : \varphi \to \psi}{\begin{array}{c}\mathbf{T} : \varphi \\ \mathbf{F} : \psi\end{array}}$$

$$\frac{\mathbf{T} : G\varphi}{\begin{array}{c}\mathbf{T} : \varphi \\ \mathbf{T} : XG\varphi\end{array}} \qquad \frac{\mathbf{F} : G\varphi}{\mathbf{F} : \varphi \quad\quad \mathbf{F} : XG\varphi} \qquad \frac{\mathbf{T} : F\varphi}{\mathbf{T} : \varphi \quad\quad \mathbf{T} : XF\varphi} \qquad \frac{\mathbf{F} : F\varphi}{\begin{array}{c}\mathbf{F} : \varphi \\ \mathbf{F} : XF\varphi\end{array}}$$

$$\frac{\mathbf{T} : \varphi\, U\, \psi}{\mathbf{T} : \psi \quad\quad \begin{array}{c}\mathbf{T} : \varphi \\ \mathbf{T} : X(\varphi\, U\, \psi)\end{array}} \qquad \frac{\mathbf{F} : \varphi\, U\, \psi}{\begin{array}{c}\mathbf{F} : \psi \\ \mathbf{F} : \varphi \quad\quad \mathbf{F} : X(\varphi\, U\, \psi)\end{array}}$$

If the node is poised, you may step:

$$\frac{\mathbf{T} : X\varphi_1 \quad \cdots \quad \mathbf{T} : X\varphi_m \quad \mathbf{F} : X\psi_1 \quad \cdots \quad \mathbf{T} : X\psi_n}{\begin{array}{c}\Downarrow \\ \mathbf{T} : \varphi_1, \cdots, \mathbf{T} : \varphi_m, \mathbf{F} : \psi_1, \cdots, \mathbf{F} : \psi_n\end{array}}$$

- **Loop rule:** If we call our current node n, and it is poised, and

– there is a node $l < n$ such that all base case and X-formulas of n were already in l;

– and for every X-eventuality - respectively $\mathbf{T} : XF\varphi$, $\mathbf{T} : X(\varphi\, U\, \psi)$, or $\mathbf{F} : XG\varphi$ - in l we have, respectively, $\mathbf{T} : \varphi$, $\mathbf{T} : \psi$, or $\mathbf{F} : \varphi$, in some node m such that $l < m \leq n$;

then the current branch terminates open (satisfiable).

- **Simple repetition rule:** If we call our current node n, and it is poised, and

  – there is a node $l < n$ with the same base case and X-formulas as n, and this includes at least one X-eventuality;

  – and there is no X-eventuality - respectively $\mathbf{T} : XF\varphi$, $\mathbf{T} : X(\varphi\, U\, \psi)$, or $\mathbf{F} : XG\varphi$ - such that, respectively, $\mathbf{T} : \varphi$, $\mathbf{T} : \psi$, or $\mathbf{F} : \varphi$, is in a node m such that $l < m \leq n$;

  then the current branch should be closed (unsatisfiable) with a cross.

1. For each of the following sequents, use tableaux to either show that they are valid by crossing every branch, or show that are they invalid by finding an open terminated branch. You do not need to explore the tableau further if you have found a terminated open branch. If the sequent is not valid, draw a diagram presenting the model that you extract from your tableau.

   For these questions, you will not need the loop and repetition rules.

   - $XFp \vdash Fp$
     Solution.

     (1) $\mathbf{T} : XFp$

     (2) $\mathbf{F} : Fp$ ✓

     (3) $\mathbf{F} : p$ from (2)

     (4) $\mathbf{F} : XFp$ from (2)

     $\times\ (1, 4)$

   - $\bot\, U\, p \vdash p$
     Solution.

     (1) $\mathbf{T} : \bot\, U\, p$ ✓

     (2) $\mathbf{F} : p$

(3) $\mathbf{T} : p$ from (1)	(4) $\mathbf{T} : \bot$ from (1)
$\times\ (2, 3)$	(5) $\mathbf{T} : X(\bot\, U\, p)$ from (1)
	$\times\ (4)$

   - $p \vdash \bot\, U\, p$
     Solution.

     (1) $\mathbf{T} : p$

     (2) $\mathbf{F} : \bot\, U\, p$ ✓

     (3) $\mathbf{F} : p$ from (2)

(4) $\mathbf{F} : \bot$ from (2)	(5) $\mathbf{F} : X(\bot\, U\, p)$ from (2)
$\times\ (1, 3)$	$\times\ (1, 3)$

   - $p \to q, Xp \vdash Xq$
     Solution.

     (1) $\mathbf{T} : p \to q$ ✓

     (2) $\mathbf{T} : Xp$ ✓

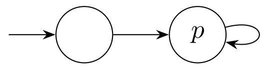
     (3) $\mathbf{F} : Xq$ ✓

(4) $\mathbf{F} : p$ from (1)	(5) $\mathbf{T} : q$ from (1)
⇟	⇟

(6) $\mathbf{T} : p$ from (2)   (8) $\mathbf{T} : p$ from (2)

(7) $\mathbf{F} : q$ from (3)   (9) $\mathbf{F} : q$ from (3)

We did not need to explore both branches, as one terminated open branch suffices, so we could have left unexplored either the branch with (5), (8), and (9), or that with (4), (6), and (7). Supposing we only explored the left hand branch, we extract the model:



We can label the first state q, or not, as we like; both choices are compatible with the left hand branch, which does not give a sign to q in the first node.

- $p, Xp \vdash Gp$

Solution.

(1) $\mathbf{T} : p$

(2) $\mathbf{T} : Xp$ ✓

(3) $\mathbf{F} : Gp$ ✓

(4) $\mathbf{F} : XGp$ from (4) ✓

⇟

(5) $\mathbf{T} : p$ from (2)

(6) $\mathbf{F} : Gp$ from (4) ✓

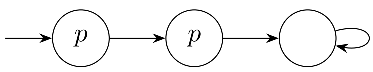
(7) $\mathbf{F} : XGp$ from (6) ✓

⇟

(8) $\mathbf{F} : Gp$ from (7) ✓

(9) $\mathbf{F} : p$ from (8)

We gave you the hint that the loop and simple repetition rules would not be needed, but it is worth understanding why they are not applicable. In fact the second node does satisfy the first condition for the loop rule - the set of base cases and X-propositions of the second node are a subset of those of the first. But the second condition of the loop rule is not satisfied, as we have the X-eventuality $\mathbf{F} : XGp$ in the first node, but no $\mathbf{F} : p$ in the second. The simple repetition rule is also not applicable, because that requires the set of base cases and X-propositions of both nodes be identical.



- $p\,U\,q, q\,U\,r \vdash p\,U\,r$

Solution.

(1) $\mathbf{T} : p\,U\,q$ ✓

(2) $\mathbf{T} : q\,U\,r$ ✓

(3) $\mathbf{F} : p\,U\,r$ ✓

(4) $\mathbf{T} : q$ from (1)

(5) $\mathbf{T} : q$ from (2)

(6) $\mathbf{T} : X(q\,U\,r)$ from (2) ✓

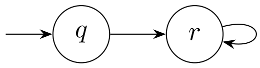
(7) $\mathbf{F} : r$ from (3)

(8) $\mathbf{F} : p$ from (3)

⇟

(9) $\mathbf{T} : q\,U\,r$ from (6)

(10) $\mathbf{T} : r$ from (9)



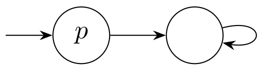
2. **The test at the start of the next tutorial will resemble this question, except that you will not be asked to explicitly extract a satisfying model where one exists.**

   Perform the same task for each of the following sequents, which will require the loop or simple repetition rules. If you find parts of your tableau are repeating parts you have already constructed, you explain in English what you conclude from that. rather than repeating yourself.

   - $Fp \vdash FXp$
     Solution.

$$(1)\ \mathbf{T} : Fp\ \checkmark$$
$$(2)\ \mathbf{F} : FXp\ \checkmark$$
$$(3)\ \mathbf{F} : Xp \text{ from } (2)\ \checkmark$$
$$(4)\ \mathbf{F} : XFXp \text{ from } (2)\ \checkmark$$
$$(5)\ \mathbf{T} : p \text{ from } (1)$$
$$\ddagger$$
$$(6)\ \mathbf{F} : p \text{ from } (3)$$
$$(7)\ \mathbf{F} : FXp \text{ from } (4)\ \checkmark$$
$$(8)\ \mathbf{F} : Xp \text{ from } (7)\ \checkmark$$
$$(9)\ \mathbf{F} : XFXp \text{ from } (7)\ \checkmark$$
$$\ddagger$$
$$(10)\ \mathbf{F} : p \text{ from } (8)$$
$$(11)\ \mathbf{F} : FXp \text{ from } (9)\ \checkmark$$
$$(12)\ \mathbf{F} : Xp \text{ from } (11)$$
$$(13)\ \mathbf{F} : XFXp \text{ from } (11)$$
$$[LOOP, 1]$$

We could compress this proof a little by replacing lines (10)-(13) with a note that the node is identical to the previous one.



   - $FXp \vdash Fp$
     Solution.

$$(1)\ \mathbf{T} : FXp\ \checkmark$$
$$(2)\ \mathbf{F} : Fp\ \checkmark$$
$$(3)\ \mathbf{F} : p \text{ from } (2)$$
$$(4)\ \mathbf{F} : XFp \text{ from } (2)\ \checkmark$$

$(5)\ \mathbf{T} : Xp$ from $(1)\ \checkmark$	$(6)\ \mathbf{T} : XFXp$ from $(1)\ \checkmark$
$\ddagger$	$\ddagger$
$(7)\ \mathbf{F} : Fp$ from $(4)\ \checkmark$	$(11)\ \mathbf{F} : Fp$ from $(4)\ \checkmark$
$(8)\ \mathbf{T} : p$ from (5)	$(12)\ \mathbf{T} : FXp$ from $(6)\ \checkmark$
$(9)\ \mathbf{F} : p$ from (7)	$(13)\ \mathbf{F} : p$ from (11)
$(10)\ \mathbf{F} : XFp$ from (7)	$(14)\ \mathbf{F} : XFp$ from $(11)\ \checkmark$
$\times\ (8, 9)$	

$(15)\ \mathbf{T} : Xp$ from $(12)\ \checkmark$	$(16)\ \mathbf{T} : XFXp$ from (12)
$\ddagger$	$[REP, 1]$
$\times$ (identical to node starting at (7))	

- $Fp \vdash XFp$

  Solution.

  (1) $\mathbf{T} : Fp$ ✓

  (2) $\mathbf{F} : XFp$ ✓

  (3) $\mathbf{T} : p$ from (1)

  ⇟

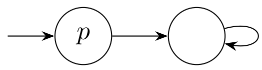
  (4) $\mathbf{F} : Fp$ from (2)

  (5) $\mathbf{F} : p$ from (4)

  (6) $\mathbf{F} : XFp$ from (4)

  ⇟

  $[LOOP, 1]$ (Identical to node above)

- $Fp, G(Xp \to p) \vdash p$

  Solution.

  (1) $\mathbf{T} : Fp$ ✓

  (2) $\mathbf{T} : G(Xp \to p)$ ✓

  (3) $\mathbf{F} : p$

  (4) $\mathbf{T} : Xp \to p$ from (2) ✓

  (5) $\mathbf{T} : XG(Xp \to p)$ from (2) ✓

  (6) $\mathbf{T} : p$ from (1) | (7) $\mathbf{T} : XFp$ from (1) ✓

  $\times\ (3, 6)$ | (8) $\mathbf{F} : Xp$ from (4) ✓ | (9) $\mathbf{T} : p$ from (4)

  ⇟ | $\times\ (3, 9)$

  $\times$ (discussed below)

  The node below the ⇟ is identical to the first node. If we take the left or right branch then we cross as per the tableau above, so we only need consider if we take the middle branch again. Then we cross with $[REP, 1]$ because the X-eventuality $\mathbf{T} : XFp$ is not paid off with a $\mathbf{T} : p$.

- $\vdash Fp\, U \neg p$

  Solution.

  (1) $\mathbf{F} : Fp\, U \neg p$ ✓

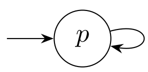
  (2) $\mathbf{F} : \neg p$ from (1) ✓

  (3) $\mathbf{F} : X(Fp\, U \neg p)$ from (1) ✓

  (4) $\mathbf{T} : p$ from (2)

  ⇟

  $[LOOP, 1]$ (Identical to node above)

- $Gp \vdash GGp$
  Solution.

(1) $\mathbf{T} : Gp$ ✓

(2) $\mathbf{F} : GGp$ ✓

(3) $\mathbf{F} : Gp$ from (2)	(4) $\mathbf{F} : XGGp$ from (2) ✓
$\times$ (1, 3)	(5) $\mathbf{T} : p$ from (1)
	(6) $\mathbf{T} : XGp$ from (1) ✓
	ǂ

The next node is identical to the first. If we choose the left branch we close as above, but if we choose the right again we close with $[REP, 1]$ because we do not make progress on the X-eventuality $\mathbf{F} : XGGp$.

- $p, XpUq \vdash F(p \wedge q)$
  Solution.

(1) $\mathbf{T} : p$

(2) $\mathbf{T} : XpUq$ ✓

(3) $\mathbf{F} : F(p \wedge q)$ ✓

(4) $\mathbf{F} : p \wedge q$ from (3) ✓

(5) $\mathbf{F} : XF(p \wedge q)$ from (3) ✓

(6) $\mathbf{F} : p$ from (4)	(7) $\mathbf{F} : q$ from (4)	
$\times$ (1, 6)	(8) $\mathbf{T} : q$ from (2)	(9) $\mathbf{T} : Xp$ from (2) ✓
	$\times$ (7, 8)	(10) $\mathbf{T} : X(XpUq)$ from (2) ✓
		ǂ

The next node is identical to the first. If we next take the left branches we cross. If we again take the rightmost branch we do not make progress on $\mathbf{T} : X(XpUq)$, so we cross with $[REP, 1]$.

- $Gp \vee Gq \vdash G(p \vee q)$
  Solution.

(1) $\mathbf{T} : Gp \vee Gq$ ✓

(2) $\mathbf{F} : G(p \vee q)$ ✓

(3) $\mathbf{T} : Gp$ from (1) ✓	(4) $\mathbf{T} : Gq$ from (1) ✓
(5) $\mathbf{T} : p$ from (3)	
(6) $\mathbf{T} : XGp$ from (3) ✓	

Under (6):

(7) $\mathbf{F} : p \vee q$ from (2) ✓	(8) $\mathbf{F} : XG(p \vee q)$ from (2) ✓
(9) $\mathbf{F} : p$ from (7)	ǂ
(10) $\mathbf{F} : q$ from (7)	(11) $\mathbf{T} : Gp$ from (6) ✓
$\times$ (5, 9)	(12) $\mathbf{F} : G(p \vee q)$ from (8) ✓
	(13) $\mathbf{T} : p$ from (11)
	(14) $\mathbf{T} : XGp$ from (11) ✓

Under (14):

(15) $\mathbf{F} : p \vee q$ from (12) ✓	(16) $\mathbf{F} : XG(p \vee q)$ from (12) ✓
(17) $\mathbf{F} : p$ from (7)	$\times$ $[REP, 1]$
(18) $\mathbf{F} : q$ from (7)	
$\times$ (13, 18)	

The tableau under (4) will be identical to that under (3), except with p replaced by q.

- $\neg q\, U\, p, \neg r\, U\, q \vdash \neg r\, U\, p$

  Solution.

(1) $\mathbf{T} : \neg q\, U\, p$ ✓

(2) $\mathbf{T} : \neg r\, U\, q$ ✓

(3) $\mathbf{F} : \neg r\, U\, p$ ✓

(4) $\mathbf{F} : p$ from (3)

(5) $\mathbf{F} : \neg r$ from (3) ✓ | (6) $\mathbf{F} : X(\neg r\, U\, p)$ from (3) ✓

(7) $\mathbf{T} : r$ from (5) | (16) $\mathbf{T} : p$ from (1) | (17) $\mathbf{T} : \neg q$ from (1)

(8) $\mathbf{T}{:}p$ from (1) | (9) $\mathbf{T}{:}\neg q$ from (1) ✓ | $\times$ (4, 16) | (18) $\mathbf{T} : X(\neg q\, U\, p)$ from (1) ✓

$\times$ (4, 8) | (10) $\mathbf{T}{:}X(\neg q\, U\, p)$ from (1) | (19) $\mathbf{F} : q$ from (17)

(11) $\mathbf{F}{:}q$ from (9) | (20) $\mathbf{T}{:}q$ from (2) | (21) $\mathbf{T}{:}\neg r$ from (2) ✓

(12) $\mathbf{T}{:}q$ from (2) | (13) $\mathbf{T}{:}\neg r$ from (2) ✓ | $\times$ (19, 20) | (22) $\mathbf{T}{:}X(\neg r\, U\, q)$ from (2) ✓

$\times$ (11, 12) | (14) $\mathbf{T}{:}X(\neg r\, U\, q)$ from (2) | (23) $\mathbf{F}{:}r$ from (21)

(15) $\mathbf{F}{:}r$ from (13) | ǂ

$\times$ (7, 15)

The next node is a copy of the first. Either we take one of the left branches and close, or take the rightmost again, in which case we close with $[REP, 1]$ because we make progress on neither eventuality $\mathbf{T} : \neg q\, U\, p, \mathbf{T} : \neg r\, U\, q$.

A further question, for consideration: can you give a straightforward meaning to the formula $\neg q\, U\, p$?

- $FGp \vdash GFp$ (Tricky)

  Solution.

(1) $\mathbf{T} : FGp$ ✓

(2) $\mathbf{F} : GFp$ ✓

(3) $\mathbf{T} : Gp$ from (1) ✓ | (4) $\mathbf{T} : XFGp$ from (1) ✓

(5) $\mathbf{T} : p$ from (3) | (19) $\mathbf{F} : Fp$ from (2) ✓ | (20) $\mathbf{F} : XGFp$ from (2) ✓

(6) $\mathbf{T} : XGp$ from (3) ✓ | (21) $\mathbf{F} : p$ from (19) | ǂ

(7) $\mathbf{F}{:}Fp$ from (2) | (8) $\mathbf{F}{:}XGFp$ from (2) ✓ | (22) $\mathbf{F} : XFp$ from (19) ✓ | see below

(9) $\mathbf{F}{:}p$ from (7) | ǂ | ǂ

(10) $\mathbf{F}{:}XFp$ from (7) | (11) $\mathbf{T}{:}Gp$ from (6) ✓ | (23) $\mathbf{T} : FGp$ from (4) ✓

$\times$ (5, 9) | (12) $\mathbf{F}{:}GFp$ from (8) ✓ | (24) $\mathbf{F} : Fp$ from (22) ✓

(13) $\mathbf{T}{:}p$ from (11) | (25) $\mathbf{F} : p$ from (24)

(14) $\mathbf{T}{:}XGp$ from (11) | (26) $\mathbf{F} : XFp$ from (24)

(15) $\mathbf{F}{:}Fp$ from (12) ✓ | (16) $\mathbf{F}{:}XGFp$ from (12) | (27) $\mathbf{T}{:}Gp$ from (23) | (28) $\mathbf{T}{:}XFGp$ from (23)

(17) $\mathbf{F}{:}p$ from (15) | $\times$ $[REP, 1]$ | (29) $\mathbf{T}{:}p$ from (27) | $\times$ $[REP, 1]$

(18) $\mathbf{F}{:}XFp$ from (15) | (30) $\mathbf{T}{:}XGp$ from (27)

$\times$ (13, 17) | $\times$ (25, 29)

The rightmost ǂ is followed by a node that is identical to the original node. If next time we take any of the left branches they will close, as per the tabelau above. So we only need consider the case where we take the rightmost branches again. But then, because the sole X-eventuality, $\mathbf{T} : XFGp$, is never paid off by a $\mathbf{T} : Gp$, $[REP, 1]$ applies, so the whole tableau closes.