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COMP3610/6361 Principles of Programming Languages

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Section 4

Proofs (Structural Induction)

Why Proofs

- how do we know that the stated theorems are actually true? intuition is often wrong – we need proof
- proofs strengthen intuition about language features
- examines all the various cases
- can guarantee items such as type safety
- most of our definitions are **inductive**; we use *structural* induction

(Mathematical) Induction

Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).

[Concrete Mathematics (1994), R. Graham]

Natural Induction I

A proof by (natural) induction consists of **two cases**.

The **base case** proves the statement for $n = 0$ without assuming any knowledge of other cases.
The **induction step** proves that if the statement holds for any given case $n = k$, then it must also hold for the next case $n = k + 1$.

Natural Induction II

Theorem

$\forall n \in \mathbb{N} . \Phi(n)$.

Proof.

Base case: show $\Phi(0)$
Induction step: $\forall k.\ \Phi(k) \Longrightarrow \Phi(k+1)$
For that we fix an arbitrary k.
Assume $\Phi(k)$ derive $\Phi(k+1)$. $\square$

Example: $0 + 1 + 2 + \cdots + n = \frac{n \cdot (n+1)}{2}$.

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Natural Induction III

Theorem

$\forall n \in \mathbb{N} . \Phi(n)$.

Proof.

Base case: show $\Phi(0)$

Induction step: $\forall i, k . 0 \leq i \leq k. \ \Phi(i) \Longrightarrow \Phi(k+1)$

For that we fix an arbitrary k.

Assume $\phi(i)$ *for all* $i \leq k$ derive $\phi(k+1)$. $\square$

Example: $F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$,

with F_n is the n-th Fibonacci number, $\varphi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $\psi = \frac{1-\sqrt{5}}{2}$.

Structural Induction I

- generalisation of natural induction
- prove that some proposition $\Phi(x)$ holds for all x of some sort of recursively defined structure
- requires well-founded partial order

Examples: lists, formulas, trees

Structural Induction II

Determinacy	structural induction for E
Progress	rule induction for $\Gamma \vdash E : T$
	(induction over the height of derivation tree)
Type Preservation	rule induction for $\langle E\, ,\, s\rangle \longrightarrow \langle E'\, ,\, s'\rangle$
Safety	mathematical induction on $\longrightarrow^n$
Uniqueness of typing	$\ldots$
Decidability of typability	exhibiting an algorithm
Decidability of type checking	corollary of other results

Structural Induction over Expressions

Prove facts about all expressions, e.g. Determinacy?

Theorem (Determinacy)

If $\langle E\,,\,s\rangle \longrightarrow \langle E_1\,,\,s_1\rangle$ *and* $\langle E\,,\,s\rangle \longrightarrow \langle E_2\,,\,s_2\rangle$
then $\langle E_1\,,\,s_1\rangle = \langle E_2\,,\,s_2\rangle$.

Do not forget the elided universal quantifiers.

Theorem (Determinacy)

For all E***,*** s***,*** E_1***,*** s_1***,*** E_2 ***and*** s_2***,***
if $\langle E\,,\,s\rangle \longrightarrow \langle E_1\,,\,s_1\rangle$ *and* $\langle E\,,\,s\rangle \longrightarrow \langle E_2\,,\,s_2\rangle$
then $\langle E_1\,,\,s_1\rangle = \langle E_2\,,\,s_2\rangle$.

Abstract Syntax

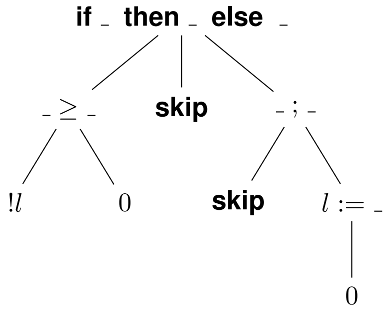
Remember the definition of expressions:

$$
\begin{aligned}
E ::= {} & n \mid b \mid E \; op \; E \mid \\
& l := E \mid \; !l \mid \\
& \textbf{if } E \textbf{ then } E \textbf{ else } E \mid \\
& \textbf{skip} \mid E \; ; E \mid \\
& \textbf{while } E \textbf{ do } E
\end{aligned}
$$

This defines an (infinite) set of expressions.

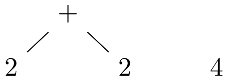
Abstract Syntax Tree I

Example: **if** $!l \geq 0$ **then skip else** ($\textbf{skip}\ ;\ l := 0$)

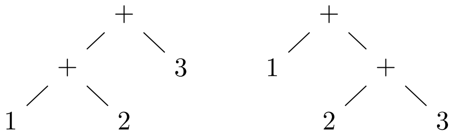


Abstract Syntax Tree II

- equivalent expressions are not the same, e.g., $2 + 2 \neq 4$



- ambiguity, e.g., $(1 + 2) + 3 \neq 1 + (2 + 3)$



Parentheses are only used for disambiguation – they are not part of the grammar

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Structural Induction (for abstract syntax)

Theorem

$\forall E \in$ *IMP*. $\Phi(E)$

Proof.

Base case(s): show $\Phi(E)$ for each unary tree constructor (leaf)
Induction step(s): show it for the remaining constructors

$$\forall c.\ \forall E_1, \ldots E_k.\ (\Phi(E_1) \wedge \cdots \wedge \Phi(E_k)) \Longrightarrow \Phi(c(E_1, \ldots, E_k))$$

$\square$

Structural Induction (syntax IMP)

To show $\forall E \in$ IMP. $\Phi(E)$.

base cases

nullary:	$\Phi(\textbf{skip})$
	$\forall b \in \mathbb{B}.\ \Phi(b)$
	$\forall n \in \mathbb{Z}.\ \Phi(n)$
	$\forall l \in \mathbb{L}.\ \Phi(\ !l)$

steps

unary:	$\forall l \in \mathbb{L}.\ \forall E.\ \Phi(E) \Longrightarrow \Phi(l := E)$
binary:	$\forall op.\ \forall E_1, E_2.\ (\Phi(E_1) \wedge \Phi(E_2)) \Longrightarrow \Phi(E_1\ op\ E_2)$
	$\forall E_1, E_2.\ (\Phi(E_1) \wedge \Phi(E_2)) \Longrightarrow \Phi(E_1\ ;\ E_2)$
	$\forall E_1, E_2.\ (\Phi(E_1) \wedge \Phi(E_2)) \Longrightarrow \Phi(\textbf{while}\ E_1\ \textbf{do}\ E_2)$
ternary:	$\forall E_1, E_2, E_3.\ (\Phi(E_1) \wedge \Phi(E_2) \wedge \Phi(E_3))$ $\Longrightarrow \Phi(\textbf{if}\ E_1\ \textbf{then}\ E_2\ \textbf{else}\ E_3)$

Proving Determinacy – Outline

Theorem (Determinacy)

For all** E**,** s**,** E_1**,** s_1**,** E_2 **and** s_2**,
if $\langle E\,,\,s\rangle \longrightarrow \langle E_1\,,\,s_1\rangle$ *and* $\langle E\,,\,s\rangle \longrightarrow \langle E_2\,,\,s_2\rangle$
then $\langle E_1\,,\,s_1\rangle = \langle E_2\,,\,s_2\rangle$.

Proof.

Choose

$$\Phi(E) \overset{\text{def}}{=} \forall s, E', s', E'', s''. \quad (\langle E\,,\,s\rangle \longrightarrow \langle E'\,,\,s'\rangle \,\wedge\, \langle E\,,\,s\rangle \longrightarrow \langle E''\,,\,s''\rangle) \Longrightarrow \langle E'\,,\,s'\rangle = \langle E''\,,\,s''\rangle$$

and show $\Phi(E)$ by structural induction over E. □

Proving Determinacy – Sketch

Some cases on whiteboard

Proving Determinacy – auxiliary lemma

Values do not reduce.

Lemma

For all $E \in$ IMP, if E is a value then
$\forall s.\ \neg(\exists E', s'.\ \langle E\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle)$.

Proof.

- E is a value iff it is of the form n, b, **skip**
- By examination of the rules . . .
  there is no rule with conclusion of the form $\langle E\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle$ for E a value

$\square$

Inversion I

In proofs involving inductive definitions. one often needs an *inversion property*.
Given a tuple in one inductively defined relation, gives you a case analysis of the possible "last rule" used.

Lemma (Inversion for $\longrightarrow$)

If $\langle E\ ,\ s\rangle \longrightarrow \langle \hat{E}\ ,\ \hat{s}\rangle$ *then either*

1. *(op$+$): there exists* n_1, n_2 *and* n *such that* $E = n_1\ op\ n_2$, $\hat{E} = n$, $\hat{s} = s$ *and* $n = n_1 + n_2$,
   (Note: $+$s have different meanings in this statements), or
2. *(op1): there exists* E_1, E_2, op *and* E_1' *such that* $E = E_1\ op\ E_2$, $\hat{E} = E_1'\ op\ E_2$ *and* $\langle E_1\ ,\ s\rangle \longrightarrow \langle E_1'\ ,\ s'\rangle$, *or*
3. *...*

Inversion II

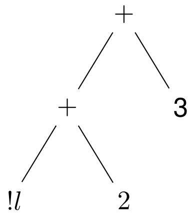
Lemma (Inversion for $\vdash$)

If $\Gamma \vdash E : T$ *then either*

- $\ldots$

Determinacy – Intuition

The intuition behind structural induction over expressions. Consider $(\,!l + 2) + 3$. How can we see that $\Phi((\,!l + 2) + 3)$ holds?



Rule Induction

How to prove the following theorems?

Theorem (Progress)

If $\Gamma \vdash E : T$ and $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$ then either E is a value or there exist E' and s' such that $\langle E\,,\,s\rangle \longrightarrow \langle E'\,,\,s'\rangle$.

Theorem (Type Preservation)

If $\Gamma \vdash E : T$*, dom*$(\Gamma) \subseteq$ *dom*(s) *and* $\langle E\,,\,s\rangle \longrightarrow \langle E'\,,\,s'\rangle$ *then* $\Gamma \vdash E' : T$ *and dom*$(\Gamma) \subseteq$ *dom*(s')*.*

Inductive Definition of $\longrightarrow$

What does $\langle E \, , \, s \rangle \longrightarrow \langle E' \, , \, s' \rangle$ actually mean?

We defined the transition relation by providing some rules, such as

(op$+$) $\quad \langle n_1 + n_2 \, , \, s \rangle \longrightarrow \langle n \, , \, s \rangle \qquad$ if $n = n_1 + n_2$

(op1) $$\frac{\langle E_1 \, , \, s \rangle \longrightarrow \langle E_1' \, , \, s' \rangle}{\langle E_1 \; op \; E_2 \, , \, s \rangle \longrightarrow \langle E_1' \; op \; E_2 \, , \, s' \rangle}$$

These rules (their concrete instances) inductively/recursively define a set of derivation trees. The last step in the derivation tree defines a step in the transition system.
We define the (infinite) set of all finite derivation trees

Derivation Tree (Transition Relation) – Example

$$\cfrac{\cfrac{\cfrac{}{\langle 2+2\,,\,\{\}\rangle \longrightarrow \langle 4\,,\,\{\}\rangle}\ (\textsc{op}+)}{\langle (2+2)+3\,,\,\{\}\rangle \longrightarrow \langle 4+3\,,\,\{\}\rangle}\ (\textsc{op}1)}{\langle (2+2)+3 \geq 5\,,\,\{\}\rangle \longrightarrow \langle 4+3 \geq 5\,,\,\{\}\rangle}\ (\textsc{op}1)$$

Derivation Tree (Typing Judgement) – Example

$$
\cfrac{
  \cfrac{
    \cfrac{\Gamma(l) = \textsf{intref}}{\Gamma \vdash !l : \textsf{int}}\ (\textsc{derref})
    \qquad
    \cfrac{}{\Gamma \vdash 2 : \textsf{int}}\ (\textsc{int})
  }{\Gamma \vdash\ !l + 2 : \textsf{int}}\ (\textsc{op+})
  \qquad
  \cfrac{}{\Gamma \vdash 3 : \textsf{int}}\ (\textsc{int})
}{\Gamma \vdash (!l + 2) + 3 : \textsf{int}}\ (\textsc{op+})
$$

Principle of Rule Induction I

For any property $\Phi(a)$ of elements a of A, and any set of rules which define a subset S_R of A, to prove

$$\forall a \in S_R.\ \Phi(a)$$

it is enough to prove that $\{a \mid \Phi(a)\}$ is closed under the rules, i.e., for each

$$\frac{h_1 \ \ \ldots h_k}{c}$$

if $\Phi(h_1) \wedge \cdots \wedge \Phi(h_k)$ then $\Phi(c)$.

Principle of Rule Induction II

For any property $\Phi(a)$ of elements a of A, and any set of rules which define a subset S_R of A, to prove

$$\forall a \in S_R.\ \Phi(a)$$

it is enough to prove that for each

$$\frac{h_1 \quad \dots h_k}{c}$$

if $\Phi(h_1) \wedge \cdots \wedge \Phi(h_k)$ then $\Phi(c)$.

Proving Progress I

Theorem (Progress)

If $\Gamma \vdash E : T$ and $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$ then either E is a value or there exist E' and s' such that $\langle E\,,\ s\rangle \longrightarrow \langle E'\,,\ s'\rangle$.

Proof.

Choose

$$\begin{aligned}\Phi(\Gamma, E, T) = \ & \forall s.\ \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s) \\ & \qquad \Longrightarrow \mathsf{value}(E) \vee (\exists E', s'.\ \langle E\,,\ s\rangle \longrightarrow \langle E'\,,\ s'\rangle)\end{aligned}$$

We show that for all Γ, E, T, if $\Gamma \vdash E : T$ then $\Phi(\Gamma, E, T)$, by rule induction on the definition of $\vdash$. □

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Proving Progress II

Rule induction for our typing rules means:

(int) $\forall \Gamma, n.\ \Phi(\Gamma, n, \mathsf{int})$

(deref) $\forall \Gamma, l.\ \Gamma(l) = \mathsf{intref} \Longrightarrow \Phi(\Gamma, !l, \mathsf{int})$

(op+) $\forall \Gamma, E_1, E_2.\ \big(\Phi(\Gamma, E_1, \mathsf{int}) \wedge \Phi(\Gamma, E_2, \mathsf{int}) \wedge \Gamma \vdash E_1 : \mathsf{int} \wedge \Gamma \vdash E_2 : \mathsf{int}\big)$
$\Longrightarrow \Phi(\Gamma, E_1 + E_2, \mathsf{int})$

(seq) $\forall \Gamma, E_1, E_2.\ \big(\Phi(\Gamma, E_1, \mathsf{unit}) \wedge \Phi(\Gamma, E_2, T) \wedge \Gamma \vdash E_1 : \mathsf{unit} \wedge \Gamma \vdash E_2 : T\big)$
$\Longrightarrow \Phi(\Gamma, E_1; E_2, \mathsf{int})$

…[10 rules in total]

Proving Progress III

$$\begin{aligned}\Phi(\Gamma, E, T) = \ & \forall s.\ \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s) \\ & \qquad \Longrightarrow \mathsf{value}(E) \vee (\exists E', s'.\ \langle E\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle)\end{aligned}$$

Case (op$+$):

$$\text{(op+)} \quad \frac{\Gamma \vdash E_1 : \mathsf{int} \qquad \Gamma \vdash E_2 : \mathsf{int}}{\Gamma \vdash E_1 + E_2 : \mathsf{int}}$$

- assume $\Phi(\Gamma, E_1, \mathsf{int})$, $\Phi(\Gamma, E_2, \mathsf{int})$, $\Gamma \vdash E_1 : \mathsf{int}$ and $\Gamma \vdash E_2 : \mathsf{int}$
- we have to show $\Phi(\Gamma, E_1 + E_2, \mathsf{int})$
- assume an arbitrary but fixed s, and $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$
- $E_1 + E_2$ is not a value; hence we have to show

$$\exists E', s'.\ \langle E_1 + E_2\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle$$

Proving Progress IV

Case (op$+$) (cont'd)**:**

- we have to show

$$\exists E', s'.\ \langle E_1 + E_2\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle$$

- Using $\Phi(\Gamma, E_1, \mathsf{int})$ and $\Phi(\Gamma, E_2, \mathsf{int})$ we have
  case E_1 reduces. Then $E_1 + E_2$ does, by (op1).
  case E_1 is a value and E_2 reduces. Then $E_1 + E_2$ does, by (op2).
  case E_1 and E_2 are values; we want to use

$$\text{(op+)} \qquad \langle n_1 + n_2\ ,\ s\rangle \longrightarrow \langle n\ ,\ s\rangle \qquad \text{if } n = n_1 + n_2$$

  we assumed $\Gamma \vdash E_1 : \mathsf{int}$ and $\Gamma \vdash E_2 : \mathsf{int}$ we need $E_1 = n_1$ and $E_2 = n_2$; then $E_1 + E_2$ reduces, by (op$+$).

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Proving Progress V

Lemma

For all Γ, E, T, *if* $\Gamma \vdash E : T$ *is a value with* $T =$ *int* *then there exists* $n \in \mathbb{Z}$ *with* $E = n$.

Derivation Tree (Typing Judgement) – Example

$$\dfrac{\dfrac{\dfrac{\Gamma(l) = \mathsf{intref}}{\Gamma \vdash\ !l : \mathsf{int}}\ (\textsc{deref}) \qquad \dfrac{}{\Gamma \vdash 2 : \mathsf{int}}\ (\textsc{int})}{\Gamma \vdash\ !l + 2 : \mathsf{int}}\ (\textsc{op+}) \qquad \dfrac{}{\Gamma \vdash 3 : \mathsf{int}}\ (\textsc{int})}{\Gamma \vdash (\ !l + 2) + 3 : \mathsf{int}}\ (\textsc{op+})$$

Which Induction Principle to Use?

- matter of convenience (all equivalent)
- use an induction principle that matches the definitions

Structural Induction (Repetition)

Determinacy	structural induction for E
Progress	rule induction for $\Gamma \vdash E : T$
	(induction over the height of derivation tree)
Type Preservation	rule induction for $\langle E\ ,\ s\rangle \longrightarrow \langle E'\ ,\ s'\rangle$
Safety	mathematical induction on $\longrightarrow^n$
Uniqueness of typing	$\ldots$
Decidability of typability	exhibiting an algorithm
Decidability of type checking	corollary of other results

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Why care about Proofs?

1. sometimes it seems hard or pointless to prove things because they are ‘obvious’, . . .
   (in particular with our language)
2. proofs illustrate (and explain) why ‘things are obvious’
3. sometimes the obvious facts are false . . .
4. sometimes the obvious facts are not obvious at all
   (in particular for ‘real’ languages)
5. sometimes a proof contains or suggests an algorithm that you need
   (proofs that type inference is decidable (for fancier type systems))
6. force a clean language design