COMP3630/6360: Theory of Computation
Semester 1, 2022
The Australian National University

Finite Automata

COMP3630/6363: Theory of Computation

Textbook.	Introduction to Automata Theory, Languages and Computation John E. Hopcroft, Rajeev Motwani, and Jeffrey D. Ullman [HMU].
Prerequisites.	Chapter 1 of HMU (sets, functions, relations, induction)
Assessment.	3 Assignments @ 10% each, Final Exam @ 70%.
Content.	Languages / Automata / Computability / Complexity.
Lecturer.	Dirk Pattinson, dirk.pattinson@anu.edu.au

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- ✓ Act as the official liaison between your classmates and your lecturers in communicating feedback about the course and providing course-related updates to your classmates. You'll also provide regular reports to the Associate Director (Education) on the feedback you've been gathering.

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This Lecture Covers Chapter 2 of HMU: Finite Automata

- Deterministic Finite Automata
- Nondeterministic Finite Automata
- NFA with ϵ-transitions
- An Equivalence among the above three.

Additional Reading: Chapter 2 of HMU.

Preliminary Concepts

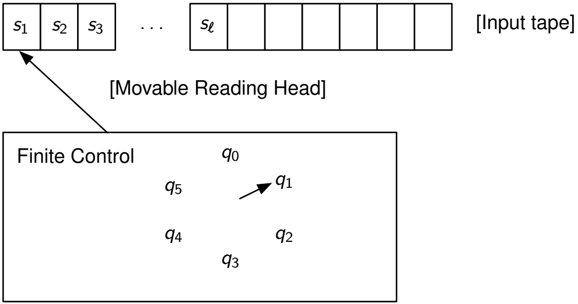
- **Alphabet** Σ: A finite set of **symbols**, e.g.,
  - $\Sigma = \{0, 1\}$ (**binary** alphabet)
  - $\Sigma = \{a, b, \ldots, z\}$ (**Roman** alphabet)
- **String** (or **word**) is a finite sequence of symbols.
  Strings are usually represented without commas, e.g., 0011 instead of $(0, 0, 1, 1)$
- **Concatenation** $x \cdot y$ of strings x and y is the string xy.
  - ϵ is the identity element for concatenation, i.e., $\epsilon \cdot x = x \cdot \epsilon = x$.
  - Concatenation of sets of strings: $A \cdot B = \{a \cdot b : a \in A, b \in B\}$
  - Concatenation of the same set: $A^2 = AA$; $A^3 = (AA)A$, etc

  (We often elide the concatenation operator and write AB for $A \cdot B$)
- Kleene-$*$ or closure operator: $A^* = \{\epsilon\} \cup A \cup A^2 \cup A^3 \cdots = \bigcup_{n \geq 0} A^n$

  (Viewing Σ as a set of length-1 strings, Σ^* is the set of all strings over Σ.)
- A **(formal) language** is a subset of Σ^*.

Deterministic Finite Automaton (DFA)

Informally:



- The device consisting of: (a) input tape; (b) reading head; and (c) finite control (Finite-state machine)
- The input is read from left to right
- Each read operation changes the internal state of the finite-state machine (FSM)
- Input is accepted/rejected based on the final state after reading all symbols

Deterministic Finite Automaton (DFA)

Definition: DFA

- A DFA $A = (Q, \Sigma, \delta, q_0, F)$
  - Q: A finite set (of internal states)
  - Σ: The alphabet corresponding to the input
  - $\delta : Q \times \Sigma \to Q$, (Transition Function)
    (If present state is $q \in Q$, and $a \in \Sigma$ is read, the DFA moves to $\delta(q, a)$.)
  - q_0: The (unique) starting state of the DFA (prior to any reading). ($q_0 \in Q$)
  - $F \subseteq Q$ is the set of final (or accepting) states

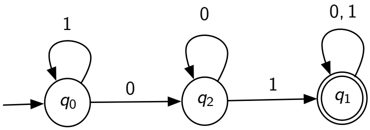
Transition Table:

	0	1
$\rightarrow q_0$	q_2	q_0
$* \ q_1$	q_1	q_1
q_2	q_2	q_1

$F = \{q_1\}$

$$\delta(q_0, 0) = q_2$$
$$\delta(q_0, 1) = q_0$$

Transition Diagram:



Language accepted by a DFA

- The language $L(A)$ accepted by a DFA $A = (Q, \Sigma, \delta, q_0, F)$ is:
  - The set of all input strings that move the state of the DFA from q_0 to a state in F
- This is formalized via the **extended** transition function $\hat{\delta} : Q \times \Sigma^* \to Q$:
- Basis:

$$\hat{\delta}(q, \epsilon) = q \quad \text{(no state change)}$$

- Induction:

$$\hat{\delta}(q, ws) = \delta(\hat{\delta}(q, w), s) \quad \text{(process } w\text{, then } s\text{)}$$

- $L(A) :=$ all strings that take q_0 to some final state $= \{w \in \Sigma^* : \hat{\delta}(q_0, w) \in F\}$.

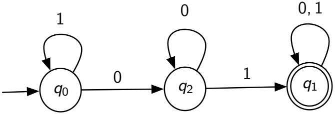
In other words:

- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

$$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

An Example

A:



- Is 00 accepted by A?
  - $q_0 \xrightarrow{0} q_2 \xrightarrow{0} q_2 \notin F$
  - Thus, $00 \notin L(A)$
- Is 001 accepted by A?
  - $q_0 \xrightarrow{0} q_2 \xrightarrow{0} q_2 \xrightarrow{1} q_1 \in F$
  - Thus, $001 \in L(A)$
- The only way one can reach q_1 from q_0 is if the string contains 01.
- $L(A)$ is the set of strings containing 01.
- Remark 1: In general, each string corresponds to a unique path of states.
- Remark 2: Multiple strings can have the same path of states. For example, 0010 and 0011 have the same sequence of states.

Limitations of DFAs

- Can all languages be accepted by DFAs?
  - DFAs have a finite number of states (and hence finite memory).
  - Given a DFA, there is always a long pattern it **cannot** 'remember' or 'track'
    - e.g., $L = \{0^n 1^n : n \in \mathbb{N}\}$ cannot be accepted by **any** DFA.
- Can generalize DFAs in one of many ways:
  - Allow transitions to multiple states for each symbol read.
  - Allow transitions without reading any symbol
  - Allow the device to have an additional tape to store symbols
  - Allow the device to edit the input tape
  - Allow bidirectional head movement

Non-deterministic Finite Automaton (NFA)

- Allow transitions to multiple states at each symbol reading.
- Multiple transitions allows the device to:
  - clone itself, traverse through and consider all possible parallel outcomes.
  - hypothesize/guess multiple eventualities concerning its input.
- Non-determinism seems bizarre, but aids in the simplification of describing an automaton.

Definition: NFA

- NFA $A = (Q, \Sigma, \delta, q_0, F)$ is defined similar to a DFA with the exception of the transition function, which takes the following form.
  - $\delta : Q \times \Sigma \to 2^Q$ (Transition Function)

- **Remark 1**: $\delta(q, s)$ can be a set with two or more states, or even be empty!
- **Remark 2**: If $\delta(\cdot, \cdot)$ is a singleton for all argument pairs, then NFA is a DFA. (So every DFA is trivally an NFA).

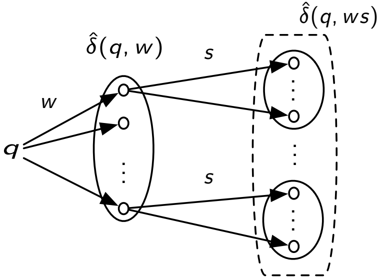
Language Accepted by an NFA

- The language accepted by an NFA is formally defined via the **extended** transition function $\hat{\delta} : Q \times \Sigma^* \to 2^Q$:
  - Basis:

    $$\hat{\delta}(q, \epsilon) = \{q\} \quad \text{(no state change)}$$

  - Induction:

    $$\hat{\delta}(q, ws) = \bigcup_{p \in \hat{\delta}(q,w)} \delta(p, s), \;\; s \in \Sigma, w \in \Sigma^*.$$



- $L(A) := \{w \in \Sigma^* : \hat{\delta}(q_0, w) \cap F \neq \emptyset\}$.

In other words:

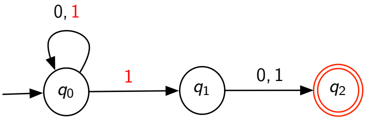
- $\epsilon \in L(A) \Leftrightarrow q_0 \in F$
- For $k > 0$,

  $$w = s_1 s_2 \cdots s_k \in L(A) \Leftrightarrow \exists \text{ a path } q_0 \xrightarrow{s_1} P_1 \xrightarrow{s_2} P_2 \xrightarrow{s_3} \cdots \xrightarrow{s_k} P_k \in F$$

An Example

- $L(A) = \{w : \text{ penultimate symbol in } w \text{ is a } 1\}$.

	0	1
$\rightarrow q_0$	q_0	$\{q_0, q_1\}$
q_1	q_2	q_2
$* q_2$	$\emptyset$	$\emptyset$



- $\hat{\delta}(q_0, 00) = \{q_0\}$ $\qquad q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0$
- $\hat{\delta}(q_0, 01) = \{q_0, q_1\}$ $\qquad q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_1 \qquad q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_0$
- $\hat{\delta}(q_0, 10) = \{q_0, q_2\}$ $\qquad q_0 \xrightarrow{1} q_0 \xrightarrow{0} q_0 \qquad q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_2$
- $\hat{\delta}(q_0, 100) = \{q_0\}$ $\qquad q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_0 \xrightarrow{0} q_0$
- An input can move the state from q_0 to q_2 only if it ends in 10 or 11.
- Each time the NFA reads a 1 (in state q_0) it considers two parallel possibilities:
  - the 1 is the penultimate symbol. (These paths die if the 1 is not actually the penultimate symbol)
  - the 1 is not the penultimate symbol.

Is Non-determinism Better?

- Non-determinism was introduced to increase the computational power.
- So is there a language L that is accepted by an NDA, but not by any DFA?

Theorem 2.4.1

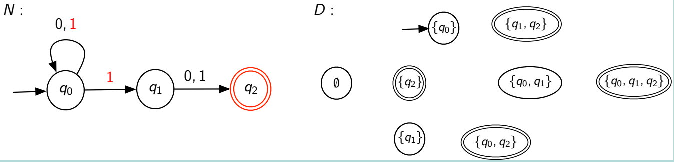
Every Language L that is accepted by an NFA is also accepted by some DFA.

Is Non-determinism Better?

Proof of Theorem 2.4.1

- Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ generate the given language L
- **Idea:** Devise a DFA D such that at any time instant the state of the DFA is the set of all states that NFA N can be in.
- Define DFA $D = (Q_D, \Sigma, \delta_D, q_{D,0}, F_D)$ from N using the following **subset construction**:

$$Q_D = 2^{Q_N} \qquad q_{D,0} = \{q_0\} \qquad F_D = \{S \subseteq Q_N : S \cap F_N \neq \emptyset\}$$



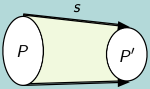
- Hence, $\epsilon \in L(N) \;\Leftrightarrow\; q_0 \in F \;\Leftrightarrow\; \{q_0\} \in F_D \;\Leftrightarrow\; \epsilon \in L(D)$

Is Non-determinism Better?

Proof of Theorem 2.4.1

- To define $\delta_D(P, s)$ for each $P \subseteq Q$ and $s \in \Sigma$:
  - Assume NFA N is simultaneously in all states of P
  - Let P' be the states to which N can transition from states in P upon reading s
  - Set $\delta_D(P, s) := P' = \bigcup_{p \in P} \delta_N(p, s)$.

N:



$\Leftrightarrow$

D : $P \xrightarrow{s} P'$

- By Induction: $\hat{\delta}_N(q_0, w) = \hat{\delta}_D(\{q_0\}, w)$ for all $w \in \Sigma^*$
  - Basis: Let $s \in \Sigma$

$$\hat{\delta}_N(q_0, \epsilon) \overset{def}{=} \{q_0\} \overset{def}{=} \hat{\delta}_D(\{q_0\})$$

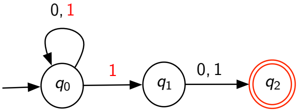
  - Induction: assume $\hat{\delta}_N(q_0, w) = \hat{\delta}_D(\{q_0\}, w)$ for $w \in \Sigma^*$

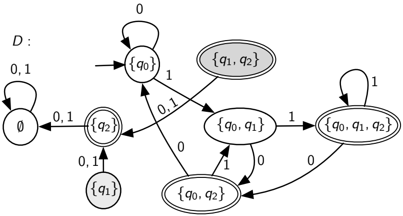
$$\hat{\delta}_N(q_0, ws) \overset{def}{=} \bigcup_{p \in \hat{\delta}_N(q_0, w)} \delta_N(p, s) \overset{ind}{=} \bigcup_{p \in \hat{\delta}_D(\{q_0\}, w)} \delta_N(p, s) \overset{def}{=} \delta_D(\hat{\delta}_D(\{q_0\}, w), s) \overset{def}{=} \hat{\delta}_D(\{q_0\}, ws)$$

- Thus, $\hat{\delta}_N(q_0, \cdot) = \hat{\delta}_D(\{q_0\}, \cdot)$, and hence the languages have to be identical.

Comments about the Subset Construction Method

- Generally, the DFA constructed using subset construction has 2^n states (n = number of states in the NFA).
- Not all states are reachable! (see example below)
- The state corresponding to the empty set is **never** a final state.





ϵ-Transitions

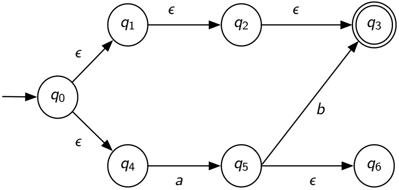
- State transitions occur without reading any symbols.

Definition: ϵ-transitions

An ϵ-Nondeterministic Finite Automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ defined similar to a DFA with the exception of the transition function, which is defined to be:

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$$

- An Example:



	ϵ	a	b
$\to q_0$	$\{q_1, q_4\}$	$\emptyset$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\emptyset$
$* q_3$	$\emptyset$	$\emptyset$	$\emptyset$
q_4	$\emptyset$	$\{q_5\}$	$\emptyset$
q_5	$\{q_6\}$	$\emptyset$	$\{q_3\}$
q_6	$\emptyset$	$\emptyset$	$\emptyset$

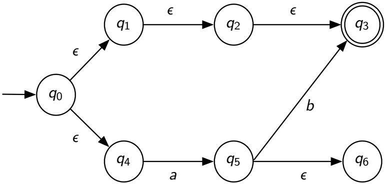
- Without reading any input symbols, the state of the ϵ-NFA can transition:

From q_0 to q_1, q_4, q_2, or q_3. From q_1 to q_2, or q_3.

From q_2 to q_3. From q_5 to q_6.

Language Accepted by an ϵ-NFA

- ϵ-**closure** of a state
  - $\texttt{ECLOSE}(q)$ = all states that are reachable from q by ϵ-transitions alone.



$$\texttt{ECLOSE}(q_0) = \{q_0, q_1, q_4, q_2, q_3\}$$
$$\texttt{ECLOSE}(q_1) = \{q_1, q_2, q_3\}$$
$$\texttt{ECLOSE}(q_2) = \{q_2, q_3\}$$
$$\texttt{ECLOSE}(q_3) = \{q_3\}$$
$$\texttt{ECLOSE}(q_4) = \{q_4\}$$
$$\texttt{ECLOSE}(q_5) = \{q_5, q_6\}$$
$$\texttt{ECLOSE}(q_6) = \{q_6\}$$

Language Accepted by an ϵ-NFA

Given ϵ-NFA $N = (Q, \Sigma, \delta, q_0, F)$ define **extended** transition function $\hat{\delta} : Q \times \Sigma^* \to 2^Q$ by induction:

- Basis:

$$\hat{\delta}(q, \epsilon) = \mathtt{ECLOSE}(q)$$

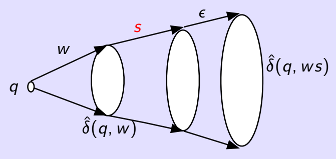
$$q \xrightarrow{\epsilon} q_1 \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} q' \qquad \epsilon = \epsilon^2 = \epsilon^3 = \cdots$$

$$\hat{\delta}(q, s) = \bigcup_{p \in \mathtt{ECLOSE}(q)} \left(\bigcup_{p' \in \delta(p,s)} \mathtt{ECLOSE}(p') \right) \qquad [s = \underbrace{\epsilon \cdots \epsilon}_{\text{finitely many}} \; s \; \underbrace{\epsilon \cdots \epsilon}_{\text{finitely many}}]$$

$$q \xrightarrow{\epsilon} q_1 \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} q' \xrightarrow{\color{red}{s}} p' \xrightarrow{\epsilon} p_1 \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} p$$

- Induction:

$$\hat{\delta}(q, ws) = \bigcup_{p \in \hat{\delta}(q,w)} \left(\bigcup_{p' \in \delta(p,s)} \color{red}{\mathtt{ECLOSE}}(p') \right)$$



- w $\in L(N)$ if and only if $\hat{\delta}(q_0, \mathrm{w}) \cap F \neq \emptyset$

Language Accepted by an ϵ-NFA

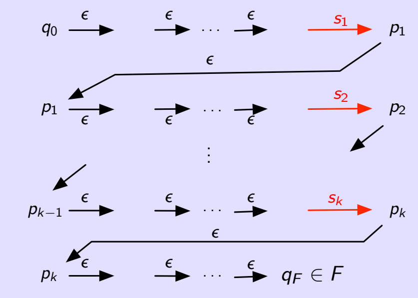
- $\mathrm{w} \in L(N)$ if and only if $\hat{\delta}(q_0, \mathrm{w}) \cap F \neq \emptyset$
- In other words:
  - $\epsilon \in L(N) \Leftrightarrow \mathtt{ECLOSE}(q_0) \cap F \neq \emptyset$

$$q_0 \xrightarrow{\epsilon} p_1 \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} p_r \in F$$

  - For $k > 0$,

∃ a path such as the following

$$w = s_1 s_2 \cdots s_k \in L(A) \quad \Leftrightarrow$$



Do ϵ-NFAs Accept More Languages?

Theorem 2.5.1

Every Language L that is accepted by an ϵ-NFA is also accepted by some DFA.

Proof of Theorem 2.5.1

- Given L that is accepted by some ϵ-NFA, we must find an NFA that accepts L. ([NFA to DFA conversion can then be done as in Theorem 2.4.1].
- Let ϵ-NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ accept L.
- Let us devise NFA $N' = (Q_{N'}, \Sigma, \delta_{N'}, q_0', F_{N'})$ as follows:

$$Q_{N'} = Q_N \qquad q_0' = q_0 \qquad F_N' = \{q \in Q_N : \mathtt{ECLOSE}(q) \cap F_N \neq \emptyset\}$$

$$\delta_{N'} : Q_{N'} \times \Sigma \to 2^{Q_{N'}} \text{ defined by: } \quad \delta_{N'}(q, s) = \bigcup_{p \in \mathtt{ECLOSE}(q)} \delta(p, s)$$

$$N : \quad q \xrightarrow{\epsilon} \quad \xrightarrow{\epsilon} \cdots \xrightarrow{\epsilon} p \xrightarrow{s} p'$$

$N :$ q can transition to p' after a few ϵ-transitions, and a single read of $s \in \Sigma$.

$$\Updownarrow$$

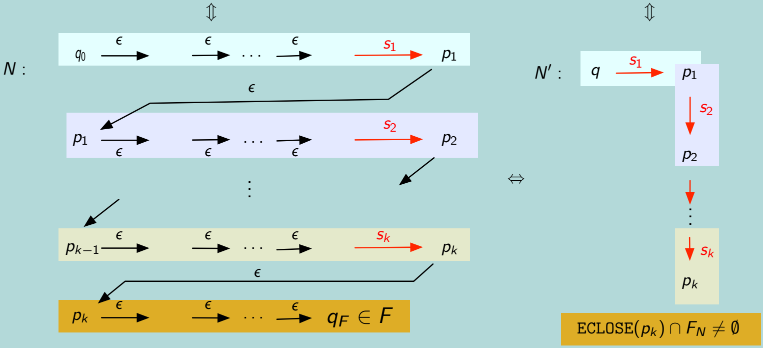
$$N' : \quad q \xrightarrow{s} p'$$

Do ϵ-NFAs Accept More Languages?

Proof of Theorem 2.5.1 (Cont'd)

[Argument is handwavy, but can be formalized!]

$s_1 \ldots s_k$ is accepted by ϵ-NFA N $\quad\Leftrightarrow\quad$ $s_1 \ldots s_k$ is accepted by NFA N'



To Summarize...

Languages accepted by DFAs	=	Languages accepted by NFAs	=	Languages accepted by ϵ-NFAs

- Allowing non-determinism and/or ϵ-transitions does not improve the language acceptance power of (finite) automata.