COMP3630/6360: Theory of Computation
Semester 1, 2022
The Australian National University

Regular Expressions

This Lecture Covers Chapter 3 of HMU: Regular Expressions and Languages

- Introduction to regular expressions and regular languages
- Equivalence of classes of regular languages and languages accepted
- Algebraic laws of (abstract) regular expressions

Additional Reading: Chapter 3 of HMU.

Regular Expressions: Overview

- So far: DFAs, NFAs were given a machine-like description
- Regular expressions are user-friendly and declarative formulation
- Regular expressions find extensive use.
  - Searching/finding strings/pattern matching or conformance in text-formatting systems (e.g., UNIX `grep`, `egrep`, `fgrep`)
  - Lexical analyzers (in compilers) use regular expressions to identify tokens (e.g., `Lex`, `Flex`)
  - In Web forms to (structurally) validate entries (passwords, dates, email IDs)
- A regular expression over an alphabet Σ is a string consisting of:
  - symbols from Σ
  - constants: $\emptyset, \epsilon$
  - operators: $+$, $*$
  - parantheses: (,)
- Each regular expression r denotes a language $L(r) \subseteq \Sigma^*$

Regular Expressions: Definition

- Regular expressions are defined inductively as follows:
  - Basis:

    B1 $\emptyset$ and ϵ are regular expressions, with $L(\emptyset) = \emptyset$ and $L(\epsilon) = \{\epsilon\}$.
    B2 For each $a \in \Sigma$, a is a regular expression with $L(a) = \{a\}$.

  - Induction: If r and s are regular expressions, then:

    I1 so is r^* with $L(r^*) = (L(r))^*$
    I2 so is $r + s$ with $L(r + s) = L(r) \cup L(s)$
    I3 so is rs with $L(r \cdot s) = L(r) \cdot L(s)$
    I4 so is (r) with $L((r)) = L(r)$.

- Only those generated by the above induction are regular.
- **Remark**: Some authors/texts use $|$ instead of $+$. HMU uses $+$.
- Precedence Rules:

$$(\cdot) > * > \cdot > +$$

where $>$ is 'binds more strongly than', and both $+$ and $\cdot$ associate to the left.

Regular Expressions: Examples

- $r = 0 + 11^*10$ is a regular expression
  - with brackets that indicate precedence: $r = 0 + (1(1^*)10)$
  - with more brackets indicating associativity: $r = 0 + ((1(1^*))1)0$
- Computing the language:

$$\begin{aligned} L(r) &= L(0) \cup L(11^*10) \\ &= \{0\} \cup L(1) \cdot L(1^*) \cdot L(1) \cdot L(0) \\ &= \{0\} \cup \{1\} \cdot \{1\}^* \cdot \{1\} \cdot \{0\} \\ &= \{0\} \cup \{1\} \cdot \{1^n \mid n \geq 0\}^* \cdot \{1\} \cdot \{0\} \\ &= \{1^i0 \mid i \neq 1\} \end{aligned}$$

- Q: What's a regular expression that describes alternating sequences of 0s and 1s?

Regular Languages: Some Basic Properties

Theorem 3.2.1

Let $w \in \Sigma^*$. Then $\{w\}$ is regular.

Proof of Theorem 3.2.1

- By induction on the length of w. For $w = \epsilon$, $\{w\} = L(\epsilon)$. For w of the form $w'x$, we have r s.t. $\{w'\} = L(r)$ so that $\{w\} = \{wx\} = L(rx)$.

Theorem 3.2.2

Let L_1 and L_2 be regular languages. Then, L_1^*, $L_1 \cup L_2$ and $L_1 L_2$ are also regular.

Proof of Theorem 3.2.2

By definition of $L(r^*)$, $L(r + s)$ and $L(rs)$.

- **Corollary 1:** The class of regular languages is closed under finite union and concatenation, i.e., if $L_1, \ldots, L_k$ are regular languages for any $k \in \mathbb{N}$, then $L_1 \cup \cdots L_k$ and $L_1 \cdots L_k$ are also regular languages.
- **Corollary 2:** Any finite language is regular.

DFAs and Regular Languages

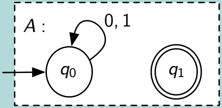
Theorem 3.2.3

For every regular language M, there exists a DFA A such that $M = L(A)$.

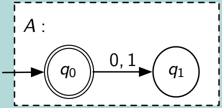
Proof of Theorem 3.2.3

- WLOG, let $\Sigma = \{0, 1\}$. Let M be a regular language. Then, $M = L(E)$ for some regular expression E.
- For each regular expression, we will devise an ϵ-NFA.
- Basis:

$\emptyset$



ϵ



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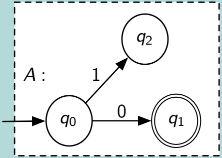

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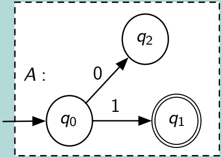
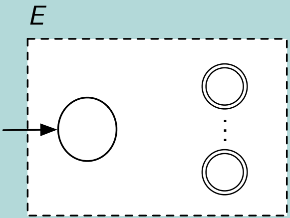
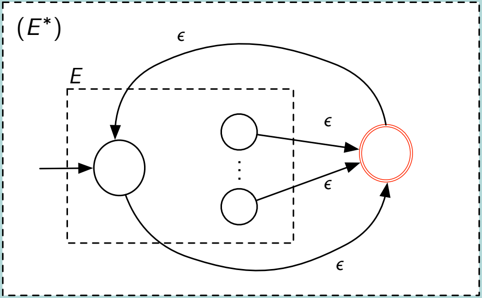


DFAs and Regular Languages

Proof of Theorem 3.2.3 (Cont'd)

- Induction E^*:

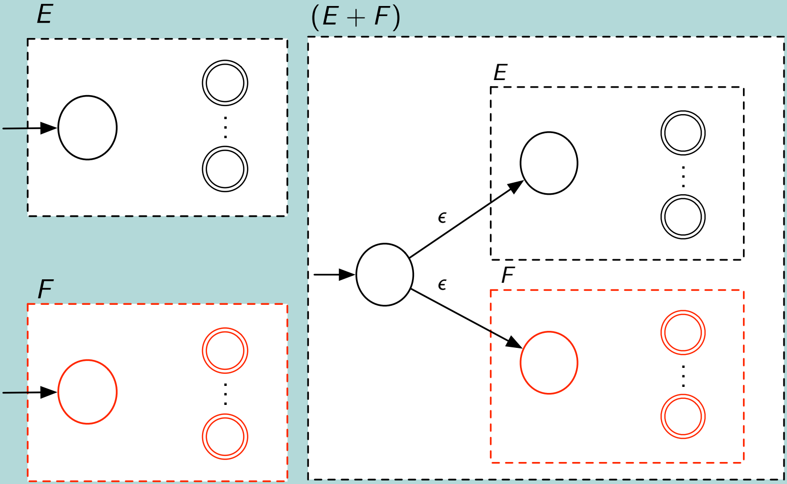




DFAs and Regular Languages

Proof of Theorem 3.2.3 (Cont'd)

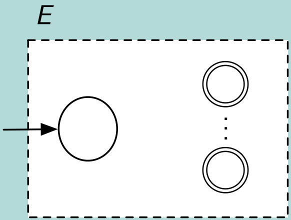
- Induction E + F:

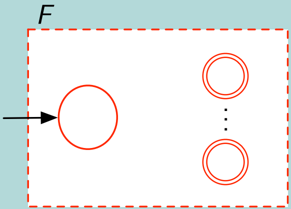


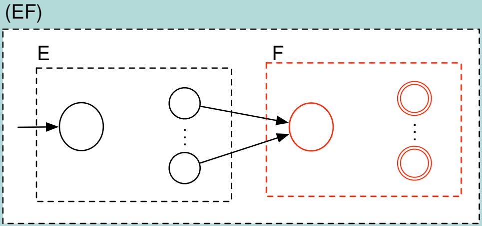
DFAs and Regular Languages

Proof of Theorem 3.2.1 (Cont'd)

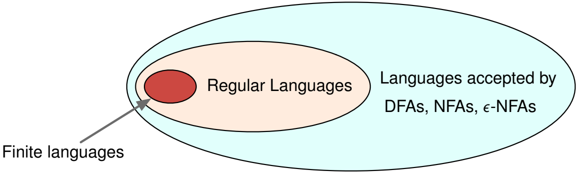
- Induction I3':







So Far...



- Is the inclusion strict?
- Are there languages accepted by DFAs that are not regular?

DFAs and Regular Languages

Theorem 3.2.4

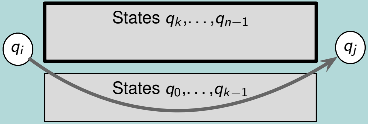
For every DFA A, there is a regular expression E such that $L(A) = L(E)$.

Proof of Theorem 3.2.4

- Let DFA $A = (Q, \Sigma, \delta, q_0, F)$ be given.
- Let us rename the states so that $Q = \{q_0, q_1, q_2, \ldots, q_{n-1})$.
- For any string $s_1 \ldots s_k \in L(A)$, there is a path

$$q_0 \xrightarrow{s_1} q_{i_1} \xrightarrow{s_2} q_{i_2} \cdots \xrightarrow{s_k} q_{i_k} \in F$$

- **Define:** $R(i, j, k)$ be the set of all input strings that move the internal state of A from q_i to q_j using paths whose intermediate nodes comprise only of q_ℓ, $\ell < k$.



- Idea: prove that (a) each $R(i, j, k)$ is regular, and (b) $L(A)$ is a union of $R(i, j, k)$'s.

DFAs and Regular Languages

Proof of Theorem 3.2.4 (Cont'd)

- Note that $L(A) = \bigcup_{j : q_j \in F} R(0, j, n)$. (i.e., paths that start in q_0 and end in an accepting state with intermediate nodes $q_0, q_1, \ldots, q_{n-1}$ (all nodes))
- $L(A)$ will be regular if each $R(i, j, k)$ to be regular. We now proceed by induction to show that each $R(i, j, k)$ is regular.
- **Basis:** Consider $R(i, j, 0)$ for $i, j \in \{0, 1, \ldots, n-1\}$.
  - $R(i, j, 0)$ consists of strings whose corresponding paths start in q_i and end in q_j with intermediate nodes q_ℓ, $\ell < 0$.

  $\Rightarrow$ No intermediate nodes

  $\Rightarrow$ $R(i, j, 0)$ contains strings that change state q_i to q_j directly

  $\Rightarrow$ $R(i, j, 0) \subseteq \{\epsilon\} \cup \Sigma$

  $\Rightarrow$ $R(i, j, 0)$ is a regular language [Corollary 2]
- **Induction:** Let $R(i, j, \ell)$ be regular for $i, j \in \{0, \ldots, n-1\}$ and $0 \leq \ell < k$. Consider $R(i, j, k)$ for $i, j \in \{0, \ldots, n-1\}$.

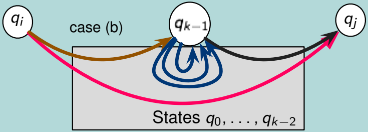
DFAs and Regular Languages

Proof of Theorem 3.2.4 (Cont'd)

- The strings in $R(i, j, k)$ correspond either to paths whose intermediate nodes belong to $\{q_0, \ldots, q_{k-1}\}$.
- Partition $R(i, j, k)$ as follows:

  Case (a): Strings whose paths **do not have** q_{k-1} as an intermediate node.

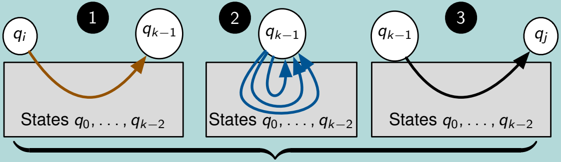
  Case (b): Strings whose paths do **pass through** q_{k-1} as an intermediate node.



- $R(i, j, k) = \{\text{Case (a) strings}\} \cup \{\text{Case (b) strings}\}$.
- Case (a) Strings are exactly those in $R(i, j, k-1)$
- Hence, $R(i, j, k) = R(i, j, k-1) \cup \{\text{Case (b) strings}\}$.

DFAs and Regular Languages

Proof of Theorem 3.2.4 (Cont'd)



- Each case (b) string is the concatenation of 3 strings:
  1. A string that changes the state from q_i to q_{k-1} through a path whose intermediate nodes are $q_0, \ldots, q_{k-2}$, i.e., $R(i, k-1, k-1)$
  2. A finite concatenation of strings, each of which take q_{k-1} back to q_{k-1} via paths that use only $q_0, \ldots, q_{k-2}$ as intermediate nodes. i.e., i.e., $R(k-1, k-1, k-1)^*$
  3. A string that takes q_{k-1} back to q_j via a path that uses only $q_0, \ldots, q_{k-2}$ as intermediate nodes, i.e., i.e., $R(k-1, j, k-1)$

  Thus,

$$R(i,j,k) = R(i,j,k-1) \cup [R(i,k-1,k-1)R(k-1,k-1,k-1)^* R(k-1,j,k-1)]$$

- From Thm 3.2.2, it follows that $R(i,j,k)$ is regular for any i,j,k. Thus, $L(A)$ is regular.

Equivalence of Languages

- The following are indeed equivalent:
  - The class of regular languages
  - The class of languages accepted by DFAs
  - The class of languages accepted by NFAs
  - The class of languages accepted by ϵ-NFAs

Properties of Regular Languages

- Regular languages are closed under finite union, concatenation, and Kleene-$*$ operation. (Theorem 3.2.2)
- They are *also* closed under:
  - Complementation: Given DFA $A = (Q, \Sigma, \delta, q_0, F)$, DFA $A' = (Q, \Sigma, \delta, q_0, F^c)$ accepts $L(A)^c$.
  - Intersection: De Morgan's Law: $R_1 \cap R_2 = (R_1^c \cup R_2^c)^c$

Abstract Regular Expressions

- We can also define **abstract** regular expressions over languages over Σ.
- Let $\mathcal{V}$ be a set of **variables** (which will be interpreted as languages)
- Use the induction definition for regular languages replacing B2 alone by:
  B2. $\mathtt{M}$ is an (abstract) regular expression for every $\mathtt{M} \in \mathcal{V}$
- **Remark:** Even though $\mathcal{V}$ could be infinite, every regular expression consists only of finitely many variables.
- Unlike **concrete** regular expressions (such as 1^*, $0 + 1$), **abstract** regular expressions (such as $\mathtt{M}^*$, $\mathtt{M} + \mathtt{N}$) don't stand for a **unique** language.
- However, we can **evaluate** abstract regular expressions by **assigning** any languages to variables, and inductively interpreting:
  - Variable* $\longrightarrow$ Kleene-$*$ closure of its language
  - Sum of variables $\longrightarrow$ union of the languages assigned to them
  - Concatenation of variables $\longrightarrow$ concatenation of their the languages
- We can introduce a notion of equality of (abstract) regular expression:

Abstract regular expressions $E_1 = E_2 \Leftrightarrow$ For any assignment of languages to the variables contained in E_1, E_2, their evaluations equal (i.e., $L(E_1) = L(E_2)$)

Algebraic Laws of Abstract Regular Expressions

- Commutativity: $\mathtt{L} + \mathtt{M} = \mathtt{M} + \mathtt{L}$ (Union is commutative)
  $\mathtt{LM} \neq \mathtt{ML}$ (Concatenation is not commutative)
- Associativity: $(\mathtt{L} + \mathtt{M}) + \mathtt{N} = \mathtt{L} + (\mathtt{M} + \mathtt{N})$ (Union is associative)
  $(\mathtt{LM})\mathtt{N} = \mathtt{L}(\mathtt{MN})$ (Concatenation is associative)
- Identity: $\emptyset + \mathtt{L} = \mathtt{L} + \emptyset = \mathtt{L}$ ($\emptyset$ is the identity element for $+$)
  $\epsilon\mathtt{L} = \mathtt{L}\epsilon = \mathtt{L}$ (ϵ is the identity element for concatenation)
- Annihilator: $\emptyset\mathtt{L} = \mathtt{L}\emptyset = \emptyset$
- Idempotent: $\mathtt{L} + \mathtt{L} = \mathtt{L}$
- Distributive: $\mathtt{L}(\mathtt{M} + \mathtt{N}) = \mathtt{LM} + \mathtt{LN}$
  $(\mathtt{M} + \mathtt{N})\mathtt{L} = \mathtt{ML} + \mathtt{NL}$
- Kleene $*$: $(\mathtt{L}^*)^* = \mathtt{L}^*$; $\quad \emptyset^* = \epsilon$; $\quad \epsilon^* = \epsilon$.