COMP3630/6360: Theory of Computation
Semester 1, 2022
The Australian National University

Context Free Languages

This lecture covers Chapter 5 of HMU: Context-free Grammars

- (Context-free) Grammars
- (Leftmost and Rightmost) Derivations
- Parse Trees
- An Equivalence between Derivations and Parse Trees
- Ambiguity in Grammars

Additional Reading: Chapter 5 of HMU.

Introduction to Grammars

- We have so far seen machine-like means (e.g., DFAs) and declarative means (e.g., regular expressions) of defining languages
- **Grammars** are a generative means of defining languages.
- Grammars can be used to create a strictly larger class of languages.
- They are especially useful in compiler and parser design; they can be used to check if:
  - parantheses are balanced in a program,
  - `else` occurrences have a matching `if`, etc.

Grammars: Formal Definition

- A **context-free** grammar (CFG) $G = (V, T, \mathcal{P}, S)$, where
  - V is a **finite** set whose elements are called **variables** or **non-terminal symbols**.
    Notation: upper case letters, e.g., $A, B, \ldots$.
  - T is a **finite** set whose elements are called **terminal symbols**; T is precisely the alphabet of the language generated by the grammar G.
    Notation: lower case letters, e.g., $s_1, s_2, \ldots$.
  - $\mathcal{P} \subseteq V \times (V \cup T)^*$ is a **finite** set of **production rules**.
    - Each production rule (A, α) is also written as $A \longrightarrow \alpha$.
      Terminology: A, α are called the head and body of the production rule, resp.
  - $S \in T$ is the unique variable/non-terminal that 'generates' the language.

Notation

- *Strings consisting of non-terminals and/or terminals will be denoted by greek symbols, e.g.,* $\alpha, \beta, \ldots$.
- *Strings of terminals will be denoted by lower case letters, e.g.,* w, u, v

How do Grammars Generate Languages?

- A string $w \in T^*$ is in the language $L(G)$ generated by $G = (V, T, \mathcal{P}, S)$ iff we can **derive** w from S, i.e.,

  start from S and use production rule(s) repeatedly to replace heads of the rules by their bodies until a string in T^* is obtained.

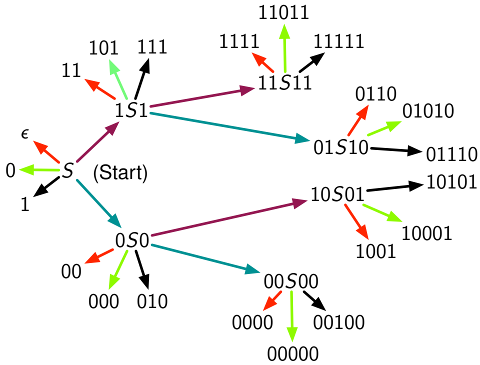
Example

Let $G = (\{S\}, \{0,1\}, \mathcal{P}, S)$ be a CFG with $\mathcal{P}$ given by

(1) $\left\{ \begin{array}{c} (S,\epsilon),(S,0),(S,1) \\ (S,0S0),(S,1S1) \end{array} \right\}$

(2)
$$\begin{array}{l} S \longrightarrow \epsilon \\ S \longrightarrow 0 \\ S \longrightarrow 1 \\ S \longrightarrow 0S0 \\ S \longrightarrow 1S1 \end{array}$$

(3) $S \longrightarrow \epsilon \,|\, 0 \,|\, 1 \,|\, 0S0 \,|\, 1S1$



Derivation: Formal Definition

Definition

Given $G = (V, T, \mathcal{P}, S)$ and $\alpha, \beta \in (V \cup T)^*$, a **derivation** *of β from α is a finite sequence of strings $\gamma_1 \underset{G}{\Rightarrow} \gamma_2 \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} \gamma_k$ for some $k \in \mathbb{N}$ where*

1. $\gamma_1 = \alpha$ *and* $\gamma_k = \beta$;
2. $\gamma_1, \ldots, \gamma_k \in (V \cup T)^*$
3. *For each $i = 1, \ldots, k-1$, γ_{i+1} is obtained from γ_i by replacing the head of a production rule of $\mathcal{P}$ by its body.*

The following phrases are used interchangeably.

$$\beta \text{ is derived from } \alpha \quad \Leftrightarrow \quad \text{there exists a derivation of } \beta \text{ from } \alpha \quad \Leftrightarrow \quad \alpha \overset{*}{\underset{G}{\Rightarrow}} \beta.$$

Example

For the grammar $G = (\{S\}, \{0,1\}, \mathcal{P}, S)$with $\mathcal{P}$ given by $S \longrightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$, the following is a derivation of 010111010 *from S*

$$S \underset{S \to 0S0}{\underset{G}{\Rightarrow}} 0S0 \underset{S \to 1S1}{\underset{G}{\Rightarrow}} 01S10 \underset{S \to 0S0}{\underset{G}{\Rightarrow}} 010S010 \underset{S \to 1S1}{\underset{G}{\Rightarrow}} 0101S1010 \underset{S \to 1}{\underset{G}{\Rightarrow}} 010111010.$$

Sentential Forms and Language Generated by a Grammar: Definitions

Definition

Given $G = (V, T, \mathcal{P}, S)$, any string in $(V \cup T)^*$ derived from S is a sentential form.

- The set of all sentential forms of G (denoted by $\mathtt{SF}(G)$) is defined inductively:
  - Basis: $S \in \mathtt{SF}(G)$
  - Induction: if $\alpha A \gamma \in \mathtt{SF}(G)$ for some $\alpha, \gamma \in (V \cup T)^*$ and $A \in V$, and $A \longrightarrow \beta$ is a production rule, then $\alpha\beta\gamma \in \mathtt{SF}(G)$.
  - Only those strings that are generated by the above induction are sentential forms.

Definition

Given CFG $G = (V, T, \mathcal{P}, S)$, the language $L(G)$ generated by G are the sentential forms that are in T^*, i.e., $L(G) = \mathtt{SF}(G) \cap T^*$.

Example

For the CFG $G = (\{S\}, \{0, 1\}, \mathcal{P}, S)$with $\mathcal{P}$ given by $S \longrightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$,

(1) S, ϵ , 0 , 1 $0S0$, 00 , 000 , 010 , $1S1$, 11 , 101 , 111 ,. . . *are all sentential forms.*

(2) $\cancel{S}$, ϵ , 0 , 1 $\cancel{0S0}$, 00 , 000 , 010 , $\cancel{1S1}$, 11 , 101 , 111 ,. . . *are in $L(G)$.*

Other Sentential Forms

- At each step of a derivation, one can replace any variable by a suitable production.
- If at each non-trivial step of the derivation the **leftmost** (or **rightmost**) variable is replaced by a production rule, then the derivation is said to be a **leftmost** (or **rightmost**) derivation, respectively. We let $\alpha \overset{*}{\underset{LM}{\Rightarrow}} \beta$ (or $\alpha \overset{*}{\underset{RM}{\Rightarrow}} \beta$) to denote the existence of a leftmost (or rightmost) derivation of β from α, respectively.
- Sentential forms derived via **leftmost** (or **rightmost**) derivations are known as **leftmost** (or **rightmost**) sentential forms, respectively.

Balanced Parantheses Example

Consider the CFG $G = (\{S\}, \{(,)\}, \mathcal{P}, S)$ with $\mathcal{P}$ given by $S \longrightarrow SS \,|\, (S) \,|\, ()$.

$$\begin{array}{rl}
[\text{Derivation}] & \underset{\uparrow}{S} \underset{G}{\Rightarrow} \underset{\uparrow}{S}S \underset{G}{\Rightarrow} (S)\underset{\uparrow}{S} \underset{G}{\Rightarrow} (\underset{\uparrow}{S})() \underset{G}{\Rightarrow} (())() \\
[\text{Leftmost Derivation}] & \underset{\uparrow}{S} \underset{G}{\Rightarrow} \underset{\uparrow}{S}S \underset{G}{\Rightarrow} (\underset{\uparrow}{S})S \underset{G}{\Rightarrow} (())\underset{\uparrow}{S} \underset{G}{\Rightarrow} (())() \\
[\text{Rightmost Derivation} & \underset{\uparrow}{S} \underset{G}{\Rightarrow} S\underset{\uparrow}{S} \underset{G}{\Rightarrow} \underset{\uparrow}{S}() \underset{G}{\Rightarrow} (\underset{\uparrow}{S})() \underset{G}{\Rightarrow} (())()
\end{array}$$

In the above, $\uparrow$ indicates the variable that is replaced in the following step

Parse Trees

- Parse trees are a graphical method of representing derivations.
- They are used in compilers to represent the source program.

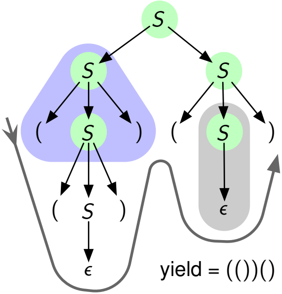
Definition

Given a CFG $G = (V, T, \mathcal{P}, S)$, a **parse tree** *for G is any directed labelled tree that meets the following three conditions:*

- *every interior node is labelled by a non-terminal (i.e., variable);*
- *every leaf node is labelled by a non-terminal, or a terminal or ϵ; however if it is labelled by ϵ, it is the sole child of its parent.*
- *if an interior node is labelled by $A \in V$, and it's children are labelled $s_1, \ldots, s_k \in V \cup T \cup \{\epsilon\}$, then $A \longrightarrow s_1 \cdots s_k$ is a production rule in $\mathcal{P}$.*

The **yield** *of a parse tree is the string formed from the labels of the tree leaves read from left to right.*
Note: *The yield is not necessarily a string of terminals.*

$$G = (\{S\}, \{(,)\}, \mathcal{P}, S)$$
$$\mathcal{P} : S \longrightarrow SS|(S)|\epsilon$$



Derivations and Parse Trees

- Parse trees, derivations, leftmost derivations, and rightmost derivations are equivalent means of generating the language $L(G)$ of a CFG G.
- The proof for equivalence of rightmost derivations mirrors that of leftmost derivations. (So we'll not delve into rightmost derivations).

Theorem 5.5.1

Let CFG $G = (V, T, \mathcal{P}, S)$ be given. Let $A \in V$ and $w \in T^*$. Then,

$$A \overset{*}{\underset{G}{\Rightarrow}} w \;\Leftrightarrow\; A \overset{*}{\underset{LM}{\Rightarrow}} w \;\Leftrightarrow\; \textit{there exists a parse tree with root } A \textit{ and yield } w \;\Leftrightarrow\; A \overset{*}{\underset{RM}{\Rightarrow}} w.$$

Proof Idea

We'll show the following implications.

$$\begin{tikzcd} & \text{Existence of a parse tree with root } A \text{ and yield } w \arrow[dl, "(b)"'] & \\ A \overset{*}{\underset{LM}{\Rightarrow}} w \arrow[rr, "\text{By Definition}"] & & A \overset{*}{\underset{G}{\Rightarrow}} w \arrow[ul, "(a)"'] \end{tikzcd}$$

Part (a) of Proof of Theorem 5.5.1: $A \overset{*}{\underset{G}{\Rightarrow}} w \Rightarrow \exists$ Parse Tree

- We use induction on the (length of the) derivation.

Lemma 5.5.2

Let CFG $G = (V, T, \mathcal{P}, S)$ be given. Let $A \in V$ and $\alpha \in SF(G)$. If $A \overset{*}{\underset{G}{\Rightarrow}} \alpha$, then there exists a parse tree with root A and yield α.

Proof of Lemma 5.5.2 (Induction on the length of derivation)

- Suppose $A \overset{*}{\underset{G}{\Rightarrow}} \alpha$ is a derivation of length 0.
- Then A is a parse tree with root A and yield A.

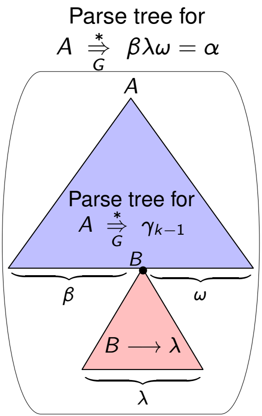
Part (a) of Proof of Theorem 5.5.1: $A \overset{*}{\underset{G}{\Rightarrow}} w \Rightarrow \exists$ Parse Tree

Proof of Lemma 5.5.2 (Induction on derivations)

- Hypothesis: the claim is true for all derivations of length $k-1$ or lesser for some $k \geq 1$.
- Suppose a derivation of α from A in k steps exists.

$$A = \gamma_1 \underset{G}{\Rightarrow} \gamma_2 \underset{G}{\Rightarrow} \gamma_3 \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} \gamma_{k-1} \underset{G}{\Rightarrow} \gamma_k = \alpha$$

- The last step must involve the application of a production rule. Hence, $\gamma_{k-1} = \beta B \omega$ and $\alpha = \beta \lambda \omega$ where (a) $\beta, \omega \in (V \cup T)^*$, (b) $B \in V$, and (b) $B \longrightarrow \lambda$ is a production rule.
- Extend the parse tree from the first $k-1$ steps by:
  - If $\lambda = X_1 \dots X_n$ with $X_1, \dots, X_n \in V \cup T$, add childen $X_1, \dots, X_n$ to node B.

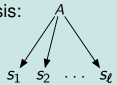


Part (b) of Proof of Theorem 5.5.1: Parse Tree $\Rightarrow A \overset{*}{\underset{LM}{\Rightarrow}} w$

Proof of Theorem 5.5.1 (Induction on the height of the tree)

- Base case: the parse tree has height 0
- Then A is a leftmost derivation in zero steps.
- Induction: Let the claim be true for all parse trees of up to height $\ell - 1$.
- Consider the root and its (say k) children. This corresponds to a production rule $A \longrightarrow X_1 \cdots X_k$.
  - If X_i is a leaf, then the yield of the sub-tree rooted at X_i is $w_i = X_i$ itself. Then trivially $X_i \overset{*}{\underset{LM}{\Rightarrow}} w_i$.
  - If X_i is not a leaf, let w_i be the yield of the parse (sub-)tree rooted at X_i of depth $\ell - 1$ or less. Then, by induction hypothesis, $X_i \overset{*}{\underset{LM}{\Rightarrow}} w_i$.

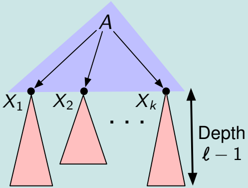
Basis:



$\alpha = s_1 \cdots s_\ell$

$(A, \alpha) \equiv (A \longrightarrow \alpha) \in \mathcal{P}$

Induction:



Then, the following is a leftmost derivation for α from A

$$A \underset{G}{\Rightarrow} {\color{red}X_1} X_2 \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} {\color{blue}w_1} {\color{red}X_2} \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} w_1 {\color{blue}w_2} {\color{red}X_3} \cdots X_k \overset{*}{\underset{LM}{\Rightarrow}} \cdots \overset{*}{\underset{LM}{\Rightarrow}} w_1 \cdots {\color{blue}w_k}$$

Ambiguity in CFGs

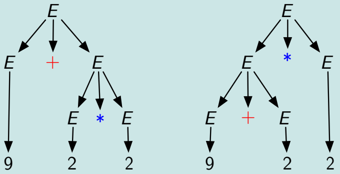
Definition

A given CFG G is **ambiguous** *if a string $w \in L(G)$ is the yield of two* **different** *parse trees. Equivalently, a CFG G is ambiguous if a string $w \in L(G)$ has two* **different** *leftmost (or rightmost) derivations.*

- Ambiguity is a property of a grammar, and **not** the language it generates.

An Example

- CFG $G = (\{E\}, \{0, 1, \ldots, 9, +, *\}, \mathcal{P}, E)$ with $\mathcal{P} : E \longrightarrow E + E | E * E | 0 | 1 | \cdots | 9$
- Consider the parse trees for $9 + 2 * 2$.
- Since there are two distinct parse trees, a compiler will not know to reduce this to 13 or to 22.



- This ambiguity is addressed by precedence rules for operators.

Ambiguity in CFGs

- Some languages are generated by unambiguous as well as ambiguous grammars.

Balanced Parantheses Example

- CFG $G_1 = (\{S\}, \{(,)\}, \mathcal{P}, S)$ with $\mathcal{P} : S \longrightarrow SS|(S)|()$
- CFG $G_2 = (\{B, R\}, \{(,)\}, \mathcal{Q}, B)$ with $\mathcal{Q} : B \longrightarrow (RB|\epsilon$ and $R \longrightarrow)|(RR$
- G_1 is ambiguous for there are two leftmost derivations for $()()()$.

$$S \underset{LM}{\Rightarrow} SS \underset{LM}{\Rightarrow} ()S \underset{LM}{\Rightarrow} ()SS \underset{LM}{\Rightarrow} ()()S \underset{LM}{\Rightarrow} ()()()$$
$$S \underset{LM}{\Rightarrow} SS \underset{LM}{\Rightarrow} SSS \underset{LM}{\Rightarrow} ()SS \underset{LM}{\Rightarrow} ()()S \overset{*}{\underset{LM}{\Rightarrow}} ()()()$$

- G_2 is **not** ambiguous, since there is precisely only one rule at any stage of derivation.

$$B \overset{*}{\underset{LM}{\Rightarrow}} (RB \underset{LM}{\Rightarrow} ()B \underset{LM}{\Rightarrow} ()(RB \underset{LM}{\Rightarrow} ()()B \underset{LM}{\Rightarrow} ()()()B \underset{LM}{\Rightarrow} ()()()\epsilon$$

- Some languages are intrinsically ambiguous, e.g., $\{0^i 1^j 2^k : i = j \text{ or } j = k\}$. All grammars for such languages are ambiguous.
- In general, there is **no** way to tell if a grammar is ambiguous.