COMP3630/6360: Theory of Computation
Semester 1, 2022
The Australian National University

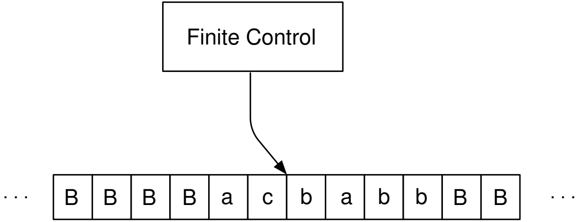
Turing Machines

This lecture covers Chapter 8 of HMU: Turing Machines

- Turing Machine
- Extensions of Turing Machines
- Restrictions of Turing Machines

Additional Reading: Chapter 8 of HMU.

Turing Machine: Informal Definition

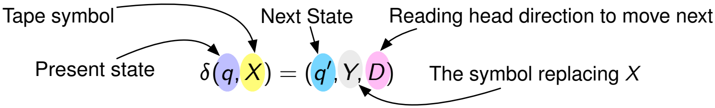


- An tape extending infinitely in both sides
- A reading head that can edit tape, move right or left.
- A finite control.
- A string is accepted if finite control reaches a final/accepting state

Turing Machine: Formal Definition

A Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ comprises of:

- Q: finite set of states
- Σ: finite set of input symbols
- Γ: finite set of tape symbols such that $\Sigma \subseteq \Gamma$
- δ: transition function. δ is a **partial function** over $Q \times \Gamma$, where the first component is viewed as the present state, and the second is viewed as the tape symbol read. If $\delta(q, X)$ is defined, then



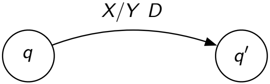
- $B \in \Gamma \setminus \Sigma$ is the blank symbol. All but a finite number of tape symbols are Bs.
- q_0: the initial state of the TM.
- F: the set of final/accepting states fo the TM.

- Head **always** moves to the left or right. Being stationary is not an option.
- The Turing Machine is deterministic.

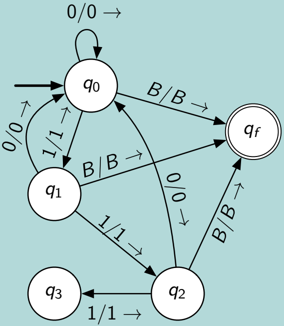
Describing TMs

- Turing machines can be defined by describing δ using a transition table.
- They can also be defined using transition diagrams (with labels appropriately altered)

If $\delta(q, X) = (q', Y, D)$

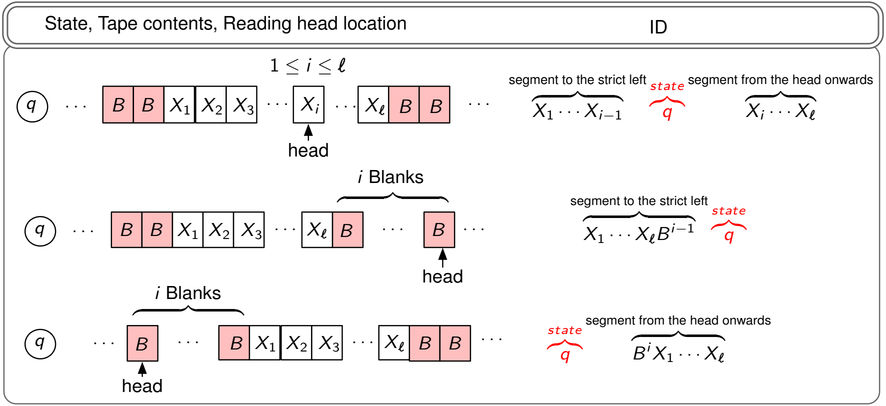


A TM that accepts any binary string that does not contain 111



Instantaneous Descriptions of TMs

- An instantaneous description (or configuration) of a TM is a complete description of the system that enables one to determine the trajectory of the TM as it operates.
- The instantaneous description or configuration or ID of a TM contains 3 parts: (a) The (finite, non-trivial) portion of tape to the left of the reading head; (b) the state that the TM is presently in; and (c) the (finite, non-trivial) portion of the tape to the right of the reading head.



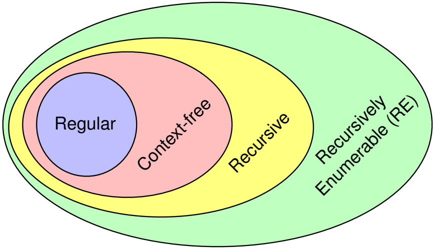
'Moves' of a TM

- Just as in the case of a PDA, we use $\underset{M}{\vdash}$ to indicate a single move of a TM M, and $\overset{*}{\underset{M}{\vdash}}$ to indicate zero or a finite number of moves of a TM.

Present ID	Transition	**Next** ID
$X_1 \cdots X_{i-1} q X_i \cdots X_\ell$ $(1 < i < \ell)$	$\delta(q, X_i) = (q', Y, R)$	$X_1 \cdots X_{i-1} Y q' X_{i+1} \cdots X_\ell$
	$\delta(q, X_i) = (q', Y, L)$	$X_1 \cdots X_{i-2} q' X_{i-1} Y X_{i+1} \cdots X_\ell$
$X_1 \cdots X_\ell B^{i-1} q$	$\delta(q, B) = (q', Y, R)$	$X_1 \cdots X_\ell B^{i-1} Y q'$
	$\delta(q, B) = (q', Y, L)$	$\begin{cases} X_1 \cdots X_{\ell-1} q' X_\ell Y & i = 1 \\ X_1 \cdots X_\ell B^{i-2} q' B Y & i > 1 \end{cases}$
$q B^i X_1 \ldots X_\ell$	$\delta(q, B) = (q', Y, R)$	$\begin{cases} Y q' X_2 \cdots X_\ell & i = 0 \\ Y q' B^{i-1} X_1 \cdots X_\ell & i > 0 \end{cases}$
	$\delta(q, B) = (q', Y, L)$	$\begin{cases} q' B Y X_2 \cdots X_\ell & i = 0 \\ q' B Y B^{i-1} X_1 \cdots X_\ell & i > 0 \end{cases}$

Language accepted by a TM

- A string w is in the language accepted by a TM M iff $q_0 w \overset{*}{\underset{M}{\vdash}} \alpha p \beta$ for some $p \in F$.
- Another notion of acceptance that is common is to require a TM to halt (i.e., no further transitions are possible).
- It is always possible to design a TM such that the TM halts when it reaches a final state without changing the language the TM accepts.
- However, we cannot require (all) TMs to halt for all inputs.
- A language L is **recursively enumerable** if it is accepted by some TM.
- A language L is **recursive** if it is accepted by a TM that **always** halts on its input.

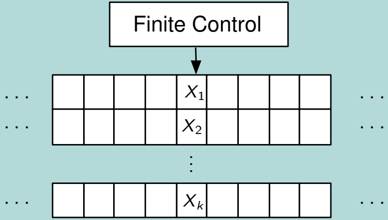


Extensions of TMs

Multiple-Track TMs

Multiple-track TM

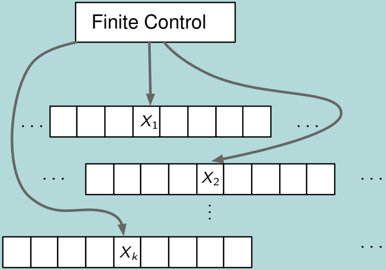
- There are k tracks, each having symbols written on them.
- The machine can only read symbols from each tape corresponding to **one** location, i.e., all symbols in a column at any one time.
- A k-track TM with tape alphabet Γ has the same langauge-acceptance power as a TM with tape alphabet Γ^k.



Multi-tape TMs

Multiple-tape TM

- There are k tapes, each having symbols written on them.
- The machine can each tape independently, i.e., the symbols read from each tape need not correspond to the same location
- After a read of each tapes, each reading head can move independently to the right, left, or stay stationary.



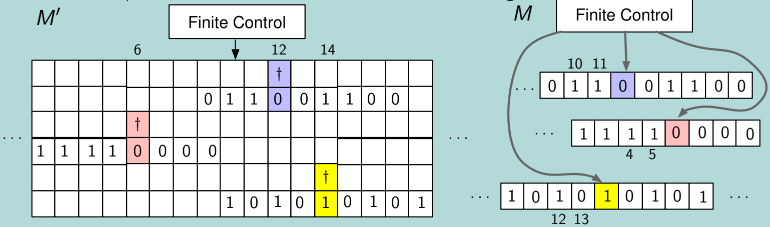
Multi-tape TMs

Theorem 7.1.1

Every language that is accepted by a multi-tape TM is also recursively enumerable (i.e., accepted by some 'standard' TM).

Proof of Theorem 7.1.1

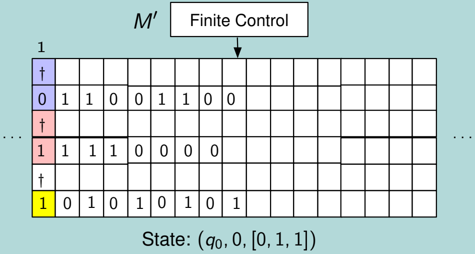
- Let L be accepted by a k-tape TM M. We'll devise a $2k$-track TM M' that accepts L.
- Every even tape of M' has the same alphabet as that of the k-tape TM. The $2i^{\text{th}}$ track of M' contains exactly the same contents as the i^{th} tape of M.
- Every odd track has an alphabet $\{B, \dagger, \}$, and contains a single $\dagger$. The $2i - 1^{\text{th}}$ track of M' contains $\dagger$ at the location where the i^{th} reading head of M is located.

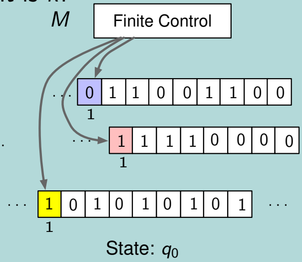


Multi-tape TMs

Proof of Theorem 7.1.1

- The state of M' has 3 components: (a) the state of M; (b) the number of †s to its strict left; and (c) a vector of length k with each component taking value in $\Gamma \cup \{?\}$.
- Each move of M takes multiple moves of M', and is a sweep of the tape from the location of the leftmost † to that of the rightmost † and back performing the changes to tracks that M would do to its corresponding tapes.
- At the beginning of the sweep, the head of M' is at a location where the leftmost † is and the state of M' is $(q, 0, [?, \cdots, ?])$. The head moves to the right uncovering †s and the corresponding track symbols (are stored in the third component of the state).
- The right sweep ends when the second component is k.





Multi-tape TMs

Proof of Theorem 7.1.1

- At this stage, M' knows the input symbols M will have read, and knows what actions to take.
- It then sweeps left making appropriate changes to the tracks (just like M does to its tape) each time a † is encountered. M' also moves the †s accordingly.
- The left sweep ends when the second component is zero. At this time, M' would have completed moving the †s and the track contents; they'll now match those of M.
- M' then moves the state to $(q', 0, [?, \cdots, ?])$ and start the next sweep if q' is not a final state.
- Note that M' mimics M and hence the languages accepted are identical.

Multi-tape TMs

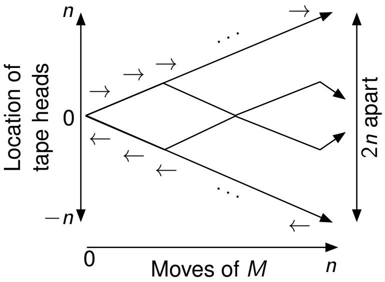
- The running time of a TM M with input w is the number of moves M makes before it halts. (If it does not, the running time is ∞).
- The time complexity $T_M : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ of a TM M is defined as follows:
  - $T_M(n) :=$ maximum running time of M for an input w of length n symbols.

Theorem 7.1.2

The time taken for M' in Theorem 7.1.2 to process n moves of M is $O(n^2)$.

Outline of Proof of Theorem 7.1.2

- After n moves of M, any two heads of M can be at most $2n$ locations apart.
- Each sweep then requires $4n$ moves of M'.
- Each track update requires a finite number of moves. Totally, to update the tracks, $\Theta(k)$ time steps are needed.



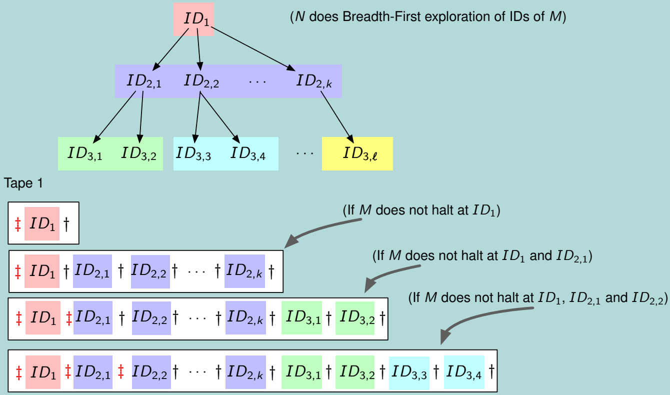
Non-deterministic TMs

Non-deterministic TM: $\delta(q, X)$ is a set of triples representing possible moves.

Theorem 7.1.3

For every non-deterministic TM M, there is a TM N such that $L(M) = L(N)$.

Outline of Proof of Theorem 7.1.3



Outline of Proof of Theorem 7.1.3

- We can devise a 2-tape TM M that simulates N.
- M first replaces the content of the first tape by ‡ followed by the ID that N is initially in, which is then followed by a special symbol †, which serves as ID separator. (M uses the second tape as scratch tape to enable this operation).
- If the ID corresponds to a final state, N halts (as would M).
- If not, M then identifies all possible choices for the next IDs for N and enters each one of them followed by † at the right end of it's first tape. (Again, M uses the second tape as scratch tape to enable this operation)
- M then searches for † to the right of ‡, changes the † to a ‡ (to signify that it is processing the succeeding ID), and processes that ID in the similar way.
- M halts at an ID iff M would at that ID.

Restrictions of TMs

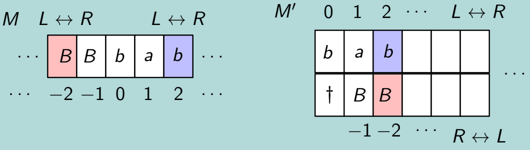
TM Semi-infinite Tape

Theorem 7.2.1

Every recursively enumerable language is also accepted by a TM with semi-infinite tape.

Outline of Proof of Theorem 7.2.1

- Given a TM M that accepts a language L, construct a two-track TM M' as follows.
- The first and second tracks of M' are the R and L semi-infinite parts of the tape of M.
- First, write a special symbol, say † at the leftmost part of the second track; this indicates to M' that a left move is not to be attempted at this location.
- At any time, M' keeps track of whether M is to the right or left of its start location.
- If M is to the strict right of its start location, M' mimics M on the first track. If M is to the strict left of its start location, M' mimics M on second track, but with the head directions reversed. M' detects the start by the † symbol.
- It can be formally shown that M' accepts a string iff M accepts it.



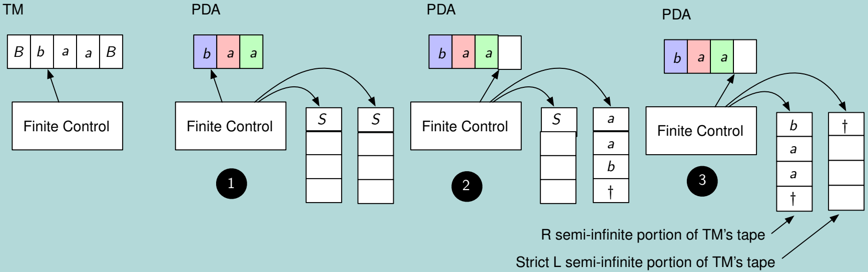
Multi-stack Machines

A multistack machne is a PDA with several independent stacks (i.e., one can be popping a symbol, while the other is writing a symbol).

Theorem 7.2.2

Every recursively enumerable language is accepted by a two-stack PDA

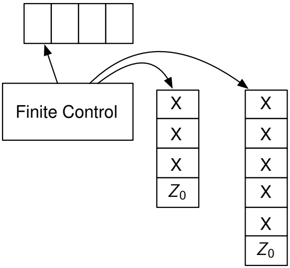
Outline of Proof of Theorem 7.2.2



- † indicates the end of the stack content (to prevent PDA from halting)
- If TM moves **right** changing tape symbol X to Y and state from q to q', PDA moves from state q to q' popping X from **left** stack and pushing Y to the **right** stack.

Counter Machines

- A counter machine is a multi-stack machine whose stack alphabet contains two symbols: Z_0 (stack end marker) and X
- Z_0 is initially in the stack.
- Z_0 may be replaced by $X^i Z_0$ for some $i \geq 0$
- X may be replaced by X^i for some $i \geq 0$.
- A counter machine effectively stores a non-negative number.



Counter Machines

Theorem 7.2.3

Every recursively enumerable language is accepted by a three-counter machine

Outline of Proof of Theorem 7.2.3

- We know a two-stack PDA can simulate any TM.
- We'll show that a 3-counter machine can simulate any (two stack) PDA.
- WLOG, let the stack alphabet of $\Gamma = \{0, 1, \ldots, r-1\}$.
- Suppose the first stack contains $Y_1(\text{top}), \ldots, Y_k$. Then the first counter stores $Y_1 + rY_2 + \cdots + r^{k-1}Y_k$. Similarly for the second stack.
- The third counter is used to change the two stack contents.
- Popping the top symbol a stack (say A) = finding quotient when $Y_1 + rY_2 + \cdots + r^{k-1}Y_k$ is divided by r.
  - pop r X's from stack A, and push a single X on the third stack. Repeat until all Xs are exhausted on the stack where popping is performed.
  - Now empty stack A and copy the third stack contents onto stack A.
- Change Y_1 to some Y_1' requires adding or subtracting, which is done by popping or pushing the corresponding number of Xs.

Counter Machines

Outline of Proof of Theorem 7.2.3

- pushing a symbol Z onto a stack (say A) = compute $rC + Z$ where C is the number presently stored in the stack A.
  - pop one X from stack A, and push r Xs on the third stack.
  - Finally push Z Xs onto the third stack. Now empty stack A and copy the third stack contents onto stack A.
- Since the above three are the only operations needed to simulate a TM on a two-stack PDA, we can stimulate a 2-stack PDA and hence a TM using a 3-counter machine.

Theorem 7.2.4

Every recursively enumerable language is accepted by a two-counter machine

Outline of Proof of Theorem 7.2.4

- The key idea: simulate three counters using one, and use the other for manipulations.
- The first counter stores $2^i 3^j 5^k$ where i, j, k are the contents of the 3-counter machine.
- Updates to the stack involve: (a) divide by 2,3, or 5; (b) multiply by 2,3, or 5; or (c) identify if i or j or k is zero (check divisibility).
- Each operation can be easily seen to be done with a spare counter.