Australian National University

COMP4011/8011
Advanced Topics in
Formal Methods and Programming Languages

– Software Verification with Isabelle/HOL –

Peter Höfner

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Section 5

Isabelle/HOL
Natural Deduction

Preview: Proofs in Isabelle

Proofs in Isabelle

General schema:

lemma name: ”<goal>”
apply <method>
apply <method>
...
done

- Sequential application of methods until all **subgoals** are solved.

The Proof State

1. $\bigwedge x_1 \ldots x_p . [\![A_1; \ldots ; A_n]\!] \Longrightarrow B$
2. $\bigwedge y_1 \ldots y_q . [\![C_1; \ldots ; C_m]\!] \Longrightarrow D$

$x_1 \ldots x_p$	Parameters
$A_1 \ldots A_n$	Local assumptions
B	Actual (sub)goal

Isabelle Theories

Syntax:

theory *MyTh*
imports $ImpTh_1 \ldots ImpTh_n$
begin
(declarations, definitions, theorems, proofs, ...)*
end

- *MyTh*: name of theory. Must live in file *MyTh*`.thy`
- $ImpTh_i$: name of *imported* theories. Import transitive.

Unless you need something special:

theory *MyTh* imports Main begin ... end

Natural Deduction Rules

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

$$\frac{A}{A \vee B}\ \frac{B}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

$$\frac{A \Longrightarrow B}{A \longrightarrow B}\ \text{impI} \qquad \frac{A \longrightarrow B \quad A \quad B \Longrightarrow C}{C}\ \text{impE}$$

For each connective ($\wedge$, $\vee$, etc):
introduction and **elimination** rules

Proof by Assumption

apply assumption

proves

1. $[\![B_1; \ldots ; B_m]\!] \Longrightarrow C$

by unifying C with one of the B_i

There may be more than one matching B_i and multiple unifiers.

Backtracking!

Explicit backtracking command: **back**

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Intro Rules

Intro rules decompose formulae to the right of $\Longrightarrow$.

apply (rule $<$intro-rule$>$)

Intro rule $[\![A_1; \dots; A_n]\!] \Longrightarrow A$ means

- To prove A it suffices to show $A_1 \dots A_n$

Applying rule $[\![A_1; \dots; A_n]\!] \Longrightarrow A$ to subgoal C:

- unify A and C
- replace C with n new subgoals $A_1 \dots A_n$

Intro Rules: example

To prove subgoal $A \longrightarrow A$ we can use: $\dfrac{P \Longrightarrow Q}{P \longrightarrow Q}$ impI

(in Isabelle: $impI$: $(?P \Longrightarrow ?Q) \Longrightarrow ?P \longrightarrow ?Q)$

Recall:
Applying rule $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$ to subgoal C:

- unify A and C
- replace C with n new subgoals $A_1 \ldots A_n$

Here:

- unify... $?P \longrightarrow ?Q$ with $A \longrightarrow A$
- replace subgoal... $A \longrightarrow A$ (i.e. $[\![\ ]\!] \Longrightarrow A \longrightarrow A$)
  with $[\![\ A\ ]\!] \Longrightarrow A$ (which can be proved with: **apply** assumption)

Elim Rules

Elim rules decompose formulae on the left of $\Longrightarrow$.

apply (erule <elim-rule>)

Elim rule $[\![A_1; \dots ; A_n]\!] \Longrightarrow A$ means

- If I know A_1 and want to prove A it suffices to show $A_2 \dots A_n$

Applying rule $[\![A_1; \dots ; A_n]\!] \Longrightarrow A$ to subgoal C:
Like **rule** but also

- unifies first premise of rule with an assumption
- eliminates that assumption

Elim Rules: example

To prove $[\![B \wedge A]\!] \Longrightarrow A$ we can use: $\dfrac{P \wedge Q \quad [\![P; Q]\!] \Longrightarrow R}{R}$ conjE

(in Isabelle: $conjE$: $[\![?P \wedge ?Q;\ [\![?P;\ ?Q]\!] \Longrightarrow ?R]\!] \Longrightarrow ?R$)

Recall:
Applying rule $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$ to subgoal C:
Like **rule** but also

- unifies first premise of rule with an assumption
- eliminates that assumption

Here:

- unify... $?R$ with A
- and also unify... $?P \wedge ?Q$ with assumption $B \wedge A$
- replace subgoal... $[\![B \wedge A]\!] \Longrightarrow A$
  with $[\![B; A]\!] \Longrightarrow A$ (which can be proved with: **apply** assumption)

Demo

More Proof Rules

Iff, Negation, True and False

$$\frac{A \Longrightarrow B \quad B \Longrightarrow A}{A = B}\ \text{iffI} \qquad \frac{A = B \quad [\![A \longrightarrow B; B \longrightarrow A]\!] \Longrightarrow C}{C}\ \text{iffE}$$

$$\frac{A = B}{A \Longrightarrow B}\ \text{iffD1} \qquad \frac{A = B}{B \Longrightarrow A}\ \text{iffD2}$$

$$\frac{A \Longrightarrow \mathit{False}}{\neg A}\ \text{notI} \qquad \frac{\neg A \quad A}{P}\ \text{notE}$$

$$\frac{}{\mathit{True}}\ \text{TrueI} \qquad \frac{\mathit{False}}{P}\ \text{FalseE}$$

Equality

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t}{t = s}\ \text{sym} \qquad \frac{r = s \quad s = t}{r = t}\ \text{trans}$$

$$\frac{s = t \quad P\ s}{P\ t}\ \text{subst}$$

Rarely needed explicitly — used implicitly by term rewriting

Classical

$$\frac{}{P = \mathit{True} \lor P = \mathit{False}} \text{ True-or-False}$$

$$\frac{}{P \lor \neg P} \text{ excluded-middle}$$

$$\frac{\neg A \Longrightarrow \mathit{False}}{A} \text{ ccontr} \qquad \frac{\neg A \Longrightarrow A}{A} \text{ classical}$$

- **excluded-middle**, **ccontr** and **classical** not derivable from the other rules.
- if we include True-or-False, they are derivable

They make the logic "classical", "non-constructive"

Cases

$$\frac{}{P \lor \neg P} \text{ excluded-middle}$$

is a case distinction on type *bool*

Isabelle can do case distinctions on arbitrary terms:

apply (case_tac *term*)

Safe and not so safe

Safe rules preserve provability

conjI, impI, notI, iffI, refl, ccontr, classical, conjE, disjE

$$\frac{A \quad B}{A \wedge B} \text{ conjI}$$

Unsafe rules can turn a provable goal into an unprovable one

disjI1, disjI2, impE, iffD1, iffD2, notE

$$\frac{A}{A \vee B} \text{ disjI1}$$

Apply safe rules before unsafe ones

Demo

What we have learned so far . . .

- natural deduction rules for $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- proof by assumption, by intro rule, elim rule
- safe and unsafe rules
- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*