Australian National University

COMP4011/8011 Advanced Topics in Formal Methods and Programming Languages

– Software Verification with Isabelle/HOL –

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Section 8

Higher Order Logic

What is Higher Order Logic?

- **Propositional Logic:**
  - no quantifiers
  - all variables have type bool
- **First Order Logic:**
  - quantification over values, but not over functions and predicates,
  - terms and formulas syntactically distinct
- **Higher Order Logic:**
  - quantification over everything, including predicates
  - consistency by types
  - formula = term of type bool
  - definition built on $\lambda^{\rightarrow}$ with certain default types and constants

Defining Higher Order Logic

Default types:

$$\text{bool} \qquad\qquad \_ \Rightarrow \_ \qquad\qquad \text{ind}$$

- **bool** sometimes called *o*
- $\Rightarrow$ sometimes called *fun*

Default Constants:

$$\begin{array}{lll} \longrightarrow & :: & bool \Rightarrow bool \Rightarrow bool \\ = & :: & \alpha \Rightarrow \alpha \Rightarrow bool \\ \epsilon & :: & (\alpha \Rightarrow bool) \Rightarrow \alpha \end{array}$$

Higher Order Abstract Syntax

Problem: Define syntax for binders like $\forall$, $\exists$, ε

One approach: $\forall$ :: *var* $\Rightarrow$ *term* $\Rightarrow$ *bool*
Drawback: need to think about substitution, α conversion again.

But: Already have binder, substitution, α conversion in meta logic

$$\lambda$$

So: Use λ to encode all other binders.

Higher Order Abstract Syntax

Example:

$$\mathrm{ALL} :: (\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$$

HOAS	**usual syntax**
$\mathrm{ALL}\ (\lambda x.\ x = 2)$	$\forall x.\ x = 2$
$\mathrm{ALL}\ P$	$\forall x.\ P\ x$

Isabelle can translate usual binder syntax into HOAS.

Side Track: Syntax Declarations

- **mixfix:**
  consts drvbl :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, _ ⊢ _")
  Legal syntax now: $\Gamma, \Pi \vdash F$
- **priorities:**
  pattern can be annotated with priorities to indicate binding strength
  Example: drvbl :: $ct \Rightarrow ct \Rightarrow fm \Rightarrow bool$ ("_, _ ⊢ _" [30, 0, 20] 60)
- **infixl/infixr**: short form for left/right associative binary operators
  Example: or :: $bool \Rightarrow bool \Rightarrow bool$ (infixr " ∨ " 30)
- **binders:** declaration must be of the form
  $c :: (\tau_1 \Rightarrow \tau_2) \Rightarrow \tau_3$ (binder "B" < p >)
  $B\ x.\ P\ x$ translated into $c\ P$ (and vice versa)
  Example ALL :: $(\alpha \Rightarrow bool) \Rightarrow bool$ (binder "∀" 10)

More in Isabelle/Isar Reference Manual (8.2)

Back to HOL

Base: $bool, \Rightarrow, ind \qquad =, \longrightarrow, \varepsilon$

And the rest is definitions:

$$
\begin{array}{lcl}
\text{True} & \equiv & (\lambda x :: bool.\ x) = (\lambda x.\ x) \\
\text{All } P & \equiv & P = (\lambda x.\ \text{True}) \\
\text{Ex } P & \equiv & \forall Q.\ (\forall x.\ P\ x \longrightarrow Q) \longrightarrow Q \\
\text{False} & \equiv & \forall P.\ P \\
\neg P & \equiv & P \longrightarrow \text{False} \\
P \wedge Q & \equiv & \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R \\
P \vee Q & \equiv & \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R \\
\text{If } P\ x\ y & \equiv & \text{SOME } z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y) \\
\text{inj } f & \equiv & \forall x\ y.\ f\ x = f\ y \longrightarrow x = y \\
\text{surj } f & \equiv & \forall y.\ \exists x.\ y = f\ x
\end{array}
$$

The Axioms of HOL

$$\frac{}{t = t}\ \text{refl} \qquad \frac{s = t \quad P\ s}{P\ t}\ \text{subst} \qquad \frac{\bigwedge x.\ f\ x = g\ x}{(\lambda x.\ f\ x) = (\lambda x.\ g\ x)}\ \text{ext}$$

$$\frac{P \Longrightarrow Q}{P \longrightarrow Q}\ \text{impI} \qquad \frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{}{(P \longrightarrow Q) \longrightarrow (Q \longrightarrow P) \longrightarrow (P = Q)}\ \text{iff}$$

$$\frac{}{P = \mathsf{True} \vee P = \mathsf{False}}\ \text{True\_or\_False}$$

$$\frac{P\ ?x}{P\ (\mathsf{SOME}\ x.\ P\ x)}\ \text{someI}$$

$$\frac{}{\exists f :: ind \Rightarrow ind.\ \mathsf{inj}\ f \wedge \neg \mathsf{surj}\ f}\ \text{infty}$$

That's it.

- 3 basic constants
- 3 basic types
- 9 axioms

With this you can define and derive all the rest.

Isabelle knows 2 more axioms:

$$\frac{x = y}{x \equiv y}\ \text{eq\_reflection} \qquad \frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Demo:
The Definitions in Isabelle

Deriving Proof Rules

In the following, we will

- look at the definitions in more detail
- derive the traditional proof rules from the axioms in Isabelle

Convenient for deriving rules: **named assumptions in lemmas**

lemma $[name :]$
assumes $[name_1 :]$ "$< prop >_1$"
assumes $[name_2 :]$ "$< prop >_2$"
$\vdots$
shows "$< prop >$" $< proof >$

proves: $[\![< prop >_1;\ < prop >_2;\ \dots]\!] \Longrightarrow < prop >$

True

consts True :: *bool*

True $\equiv (\lambda x :: bool.\ x) = (\lambda x.\ x)$

Intuition:

right hand side is always true

Proof Rules:

$$\frac{}{\text{True}}\ \text{TrueI}$$

Proof:

$$\dfrac{\dfrac{}{(\lambda x :: bool.\ x) = (\lambda x.\ x)}\ \text{refl}}{\text{True}}\ \text{unfold True\_def}$$

Demo

Universal Quantifier

consts ALL :: $(\alpha \Rightarrow \mathit{bool}) \Rightarrow \mathit{bool}$
ALL $P \quad \equiv \quad P = (\lambda x.\ \text{True})$

Intuition:

- ALL P is Higher Order Abstract Syntax for $\forall x.\ P\ x$.
- P is a function that takes an x and yields a truth value.
- ALL P should be true iff P yields true for all x, i.e. if it is equivalent to the function $\lambda x.\ \text{True}$.

Proof Rules:

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

Proof: Isabelle Demo

False

consts False :: *bool*

False $\equiv$ $\forall P. P$

Intuition:

Everything can be derived from *False*.

Proof Rules:

$$\frac{\text{False}}{P}\ \text{FalseE} \qquad \frac{}{\text{True} \neq \text{False}}$$

Proof: Isabelle Demo

Negation

consts Not :: $bool \Rightarrow bool$ ($\neg$ _)
$\neg P \equiv P \longrightarrow \mathrm{False}$

Intuition:
Try $P = True$ and $P = False$ and the traditional truth table for $\longrightarrow$.

Proof Rules:

$$\frac{A \Longrightarrow False}{\neg A}\ \mathrm{notI} \qquad \frac{\neg A \quad A}{P}\ \mathrm{notE}$$

Proof: Isabelle Demo

Existential Quantifier

consts EX :: $(\alpha \Rightarrow bool) \Rightarrow bool$
EX $P \;\equiv\; \forall Q.\, (\forall x.\, P\ x \longrightarrow Q) \longrightarrow Q$

Intuition:

- EX P is HOAS for $\exists x.\ P\ x$. (like $\forall$)
- Right hand side is characterization of $\exists$ with $\forall$ and $\longrightarrow$
- Note that inner $\forall$ binds wide: $(\forall x.\ P\ x \longrightarrow Q)$
- Remember lemma from last time:
  $(\forall x.\ P\ x \longrightarrow Q) = ((\exists x.\ P\ x) \longrightarrow Q)$

Proof Rules:

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

Proof: Isabelle Demo

Conjunction

consts And :: $bool \Rightarrow bool \Rightarrow bool$ $(\_ \wedge \_)$

$P \wedge Q \equiv \forall R.\ (P \longrightarrow Q \longrightarrow R) \longrightarrow R$

Intuition:

- Mirrors proof rules for $\wedge$
- Try truth table for P, Q, and R

Proof Rules:

$$\frac{A \quad B}{A \wedge B}\ \text{conjI} \qquad \frac{A \wedge B \quad [\![A; B]\!] \Longrightarrow C}{C}\ \text{conjE}$$

Proof: Isabelle Demo

Disjunction

consts Or :: $bool \Rightarrow bool \Rightarrow bool$ $(\_ \vee \_)$
$P \vee Q \equiv \forall R.\ (P \longrightarrow R) \longrightarrow (Q \longrightarrow R) \longrightarrow R$

Intuition:

- Mirrors proof rules for $\vee$ (case distinction)
- Try truth table for P, Q, and R

Proof Rules:

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B}\ \text{disjI1/2} \qquad \frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C}\ \text{disjE}$$

Proof: Isabelle Demo

If-Then-Else

consts If $:: bool \Rightarrow \alpha \Rightarrow \alpha \Rightarrow \alpha$ (if_ then _ else _)

If $P\ x\ y \ \equiv\ \text{SOME}\ z.\ (P = \text{True} \longrightarrow z = x) \wedge (P = \text{False} \longrightarrow z = y)$

Intuition:

- for $P = \text{True}$, right hand side collapses to $\text{SOME}\ z.\ z = x$
- for $P = \text{False}$, right hand side collapses to $\text{SOME}\ z.\ z = y$

Proof Rules:

$$\frac{}{\text{if True then } s \text{ else } t = s}\ \text{ifTrue} \qquad \frac{}{\text{if False then } s \text{ else } t = t}\ \text{ifFalse}$$

Proof: Isabelle Demo

That was HOL

We have learned ...

- Defining HOL
- Higher Order Abstract Syntax
- Deriving proof rules