Modal Logic

ANU Logic Summer School

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Introduction

(BRV: preface)

Logics

- Propositional logic: easy to deal with but weak
- Modal logics lie in between
- First-order logic: strong but difficult to deal with

Slogans

1. ML is simple yet expressive for talking about relational structures
2. ML provides an internal, local perspective on relational structures
3. Modal languages are not isolated formal systems

Question

How does modal logic relate to first-order logic?

Literature

Modal Logic by Blackburn, de Rijke & Venema (BRV)

Overview

Today

- Language and semantics of modal logic
- Other modal operators
- Correspondence

Tomorrow

- Translation into first-order logic
- Bisimulations
- Hennessy-Milner theorems

Today

- Constructing ω-saturated models
- Van Benthem characterisation theorem
- Variations

Language and semantics

Language

(BRV: §1.2)

Language *ML* $\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \qquad p \in \mathsf{Prop}$

Some formulas $\Box\top, \quad \neg(\neg\Box\neg p \wedge \neg\Box p), \quad \neg(\Box p \wedge \neg p)$

Other connectives

$$\begin{aligned} \bot &:= \neg\top \\ \varphi \vee \psi &:= \neg(\neg\varphi \wedge \neg\psi) \\ \varphi \to \psi &:= \neg\varphi \vee \psi \\ \Diamond\varphi &:= \neg\Box\neg\varphi \end{aligned}$$

More formulas(?) $\neg\Diamond\bot, \quad \Diamond p \to \Box p, \quad \Box p \to p$

Language

(BRV: §1.2)

Interpretation

$\Box\varphi =$ "φ necessarily holds"

$\Diamond\varphi =$ "φ possibly holds"

Temporal

$\Box\varphi =$ "it is henceforth the case that φ"

$\Diamond\varphi =$?

Deontic

$\Box\varphi =$ "it is obligatory that φ"

$\Diamond\varphi =$?

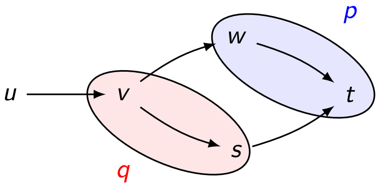
Epistemic

$\Box\varphi =$ "it is known that φ"

$\Diamond\varphi =$?

Semantics by example

(BRV: §1.3)



$W = \{u, v, w, s, t\}$

$R \subseteq W \times W$

$V(p) = \{w, t\}, V(q) = \{v, s\}$

Interpretation of $\Box$

$x \Vdash \Box\varphi$ if $\forall y.\ x \longrightarrow y$ implies $y \Vdash \varphi$

- $w \Vdash \Box p$
- $v \Vdash \Box(p \vee q)$

Exercise 1

Show that $x \Vdash \Diamond\varphi$ if $\exists y$ s.t. $x \longrightarrow y$ and $y \Vdash \varphi$

- $v \Vdash \Diamond p$
- $s \nVdash \Diamond q$

Semantics

(BRV: §1.3)

Kripke model $\mathfrak{M} = (W, R, V)$

- W: set of worlds
- R: relation
- V: valuation $V : \mathrm{Prop} \to \mathcal{P}W$

Interpretation

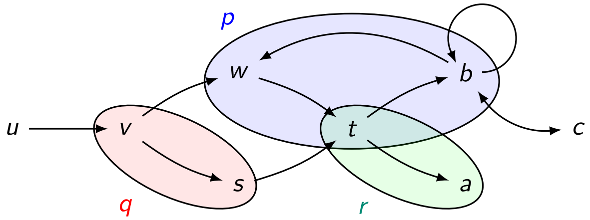
$\mathfrak{M}, w \Vdash p$	iff	$w \in V(p)$
$\mathfrak{M}, w \Vdash \top$		always
$\mathfrak{M}, w \Vdash \neg\varphi$	iff	$\mathfrak{M}, w \nVdash \varphi$
$\mathfrak{M}, w \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \Box\varphi$	iff	$\forall v \in W . wRv$ implies $\mathfrak{M}, v \Vdash \varphi$

It follows that

$\mathfrak{M}, w \Vdash \varphi \vee \psi$	iff	$\mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \varphi \to \psi$	iff	$\mathfrak{M}, w \nVdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$
$\mathfrak{M}, w \Vdash \Diamond\varphi$	iff	$\exists v \in W . wRv$ and $\mathfrak{M}, v \Vdash \varphi$

Semantics

(BRV: §1.3)



Exercise 2

For each formula, find a world where it is true and one where where it is false

(a) $\Diamond(p \vee q)$

(b) $\Box\bot$

(c) $\Diamond p \to \Diamond\Diamond p$

(d) $\Diamond\Box r$

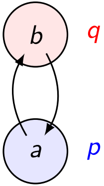
(e) $\Diamond\Diamond\Diamond\Diamond p$

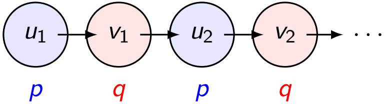
Modal equivalence

(BRV: §2.1)

Two worlds are modally equivalent if they satisfy precisely the same formulas

Example





Claim $a \leftrightsquigarrow u_i$ and $b \leftrightsquigarrow v_i$

Proof By induction on structure of φ, clear if $\varphi = p$

$\top, \neg, \wedge$ Straightforward

$\varphi = \Box\psi$ $a \Vdash \Box\psi \iff b \Vdash \psi \overset{IH}{\iff} v_i \Vdash \psi \iff u_i \Vdash \Box\psi$

Modal equivalence

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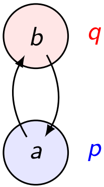
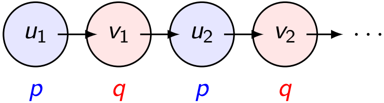
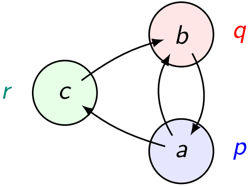




Exercise 3

Adapt the right model above such that it has a world that is modally equivalent to c



Other modal operators

Lewis' strict implication

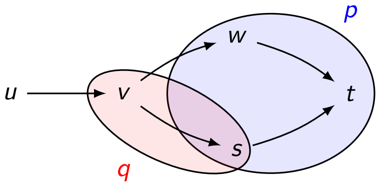
Strict implication . . . is a binary modality $\strictif$ that models our idea of implication better than $\to$

Interpretation Let $\mathfrak{M} = (W, \leq, V)$ be a Kripke model and $w \in W$, then

$$\mathfrak{M}, w \Vdash \varphi \strictif \psi \quad \text{iff} \quad \forall v \in W (\text{if } wRv \text{ and } \mathfrak{M}, v \Vdash \varphi \text{ then } v \Vdash \psi)$$

Examples

1. $v \Vdash q \strictif p$
2. $v \nVdash p \strictif q$
3. $w \Vdash (p \wedge q) \strictif \neg p$



$\Box$ versus $\strictif$

- $\Box\varphi \equiv \top \strictif \varphi$
- $\varphi \strictif \psi \equiv \Box(\varphi \to \psi)$

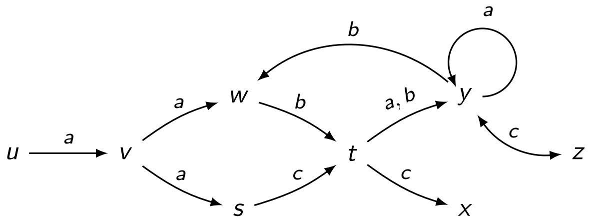
Hennessy-Milner logic

Intuition

Model actions of a program

HM logic

- No proposition letters
- Modality $\Box_a$ for every $a \in Action$

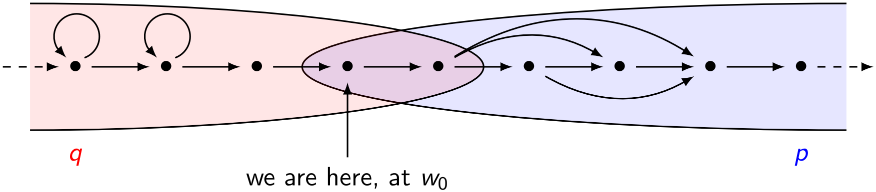


- $u \Vdash \Diamond_a \top$ means that action a can be executed
- $u \Vdash \Box_a \bot$ means that a cannot be executed

Linear temporal logic (LTL)

A timeline

Take the Kripke model $(\mathbb{Z}, \leq, V)$ with $V(p)$ and $V(q)$ as below



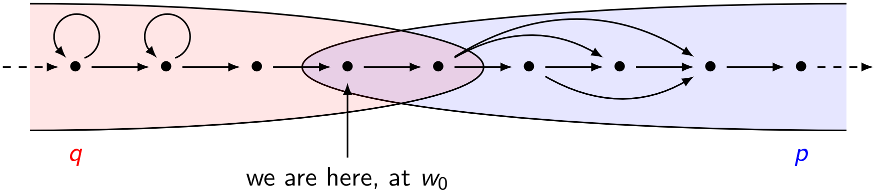
Modalities of LTL

$\Box\varphi$	"φ will always be the case"
$\Diamond\varphi$	"sometime (in the future) φ will be the case"
$\bigcirc\varphi$	"φ will be true in at the next step"
$\varphi\,\mathcal{U}\,\psi$	"φ is true until ψ is true, which must happen at some point"
$\varphi\,\mathcal{R}\,\psi$	"φ is true until ψ is true, or φ holds forever"

Tense modal logic

A timeline

Take the Kripke model $(\mathbb{Z}, \leq, V)$ with $V(p)$ and $V(q)$ as below



Looking back

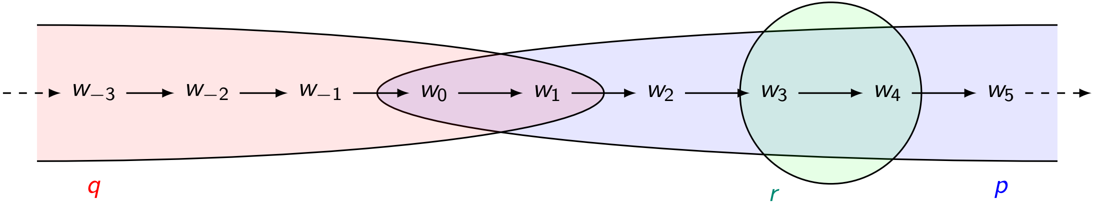
$\Box\varphi$ "φ will always be the case"

$\Diamond\varphi$ "sometime (in the future) φ will be the case"

$\blacksquare\varphi$ "φ has always been true"

$\blacklozenge\varphi$ "φ was true at some point in the past"

Tense modal logic



Let R be the reflexive transitive closure of the arrows above.

Exercise 4 Determine at which worlds the following formulas are true

(a) $\Diamond \blacklozenge \Box p$

(b) $\Diamond r \to \blacksquare p$

(c) $\Diamond r \wedge \Box p$

(d) $\Diamond (r \vee \blacksquare q)$

(e) $\blacklozenge s \to \Box \blacklozenge s$ (for any valuation of s)

Correspondence

Correspondence

(BRV: §3.1-3)

Definition A Kripke frame $\mathfrak{F} = (W, R)$ validates φ if φ is true at every world under every valuation, notation: $\mathfrak{F} \Vdash \varphi$

Proposition $\mathfrak{F} \Vdash p \to \Diamond p$ iff $\mathfrak{F}$ is reflexive (wRw for all $w \in W$)

Proof

($\Leftarrow$) If (W, R) is reflexive and $w \Vdash p$ then $w \Vdash \Diamond p$ since wRw

($\Rightarrow$) If (W, R) is not reflexive then pick $w \in W$ such that $(w, w) \notin R$. Let $V(p) = \{w\}$. Then $w \Vdash p$ but $w \nVdash \Diamond p$, so $\mathfrak{F} \nVdash p \to \Diamond p$

Correspondence

(BRV: §3.1-3)

Definition
A Kripke frame (X, R) validates φ if φ is true at every world under every valuation

Exercise 5
Find a formula $\varphi \in ML$ such that $\mathfrak{F} \Vdash \varphi$ iff (W, R) satisfies ...

- (W, R) is transitive (if uRv and vRw then uRw)
- each world has at most one successor
- each world has at least one successor

Exercise 6
Describe the class of Kripke frames satisfying ...

(a) $p \to \Diamond p$
(b) $p \to \Box\Diamond p$
(c) $\Box p \to \Diamond p$

Soundness and completeness

Axiomatisation

(BRV: §1.6 and §4)

We write $\vdash \varphi$ if the formula φ can be derived using the the following axioms and rules:

(Prop) all instances of propositional tautologies

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(MP) $$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

(Nec) $$\frac{\varphi}{\Box\varphi}$$

Theorem (soundness) $\vdash \varphi$ then φ holds in every world of every Kripke model

Theorem (completeness) If φ holds in every world of every Kripke model then $\vdash \varphi$

Proof Somewhat similar to case for first-order logic from John Slaney's course

Overview

Today	• Language and semantics of modal logic	✓
	• Other modal operators	✓
	• Correspondence	✓
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	• Bisimulations	
	• Hennessy-Milner theorems	
Today	• Constructing ω-saturated models	
	• Van Benthem characterisation theorem	
	• Variations	