# Modal Logic

ANU Logic Summer School

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# Overview

Yesterday

- Language and semantics of modal logic ✓
- Other modal operators ✓
- Correspondence ✓

Today

- Translation into first-order logic
- Bisimulations
- Hennessy-Milner theorems

Today

- Constructing $\omega$-saturated models
- Van Benthem characterisation theorem
- Variations

# The standard translation

# Kripke models revisited

The language *FOL*

One binary predicate $R$

A unary predicate $P$ for each $p \in \mathrm{Prop}$

| **Kripke model for *ML*** | **Model for *FOL*** |
|---|---|
| $\mathfrak{M} = (W, R, V)$ | $\mathcal{M} = (D_{\mathcal{M}}, R_{\mathcal{M}}, \{P_{\mathcal{M}}\})$ |
| Set of worlds | Domain |
| Binary relation $R$ | Interpretation $R_{\mathcal{M}}$ of $R$ |
| Valuation $V(p)$ of $p$ | Interpretation $P_{\mathcal{M}}$ of $P$ |

Conclusion

Kripke models coincide with models for *FOL*

# The standard translation

(BRV: §2.4)

**Notation**

- For $\varphi \in ML$, we write $\mathfrak{M}, w \Vdash \varphi$ if $\varphi$ holds at $w$
- For $\alpha \in FOL$, we write $\mathfrak{M} \models \alpha$ if $\mathfrak{M}$ is a model for $\alpha$
- If $\alpha \in FOL$ has one free variable $x$,
  then we write $\mathfrak{M} \models \alpha[w]$ if $\mathfrak{M}$ is a model for $\alpha$ with $x$ interpreted as $w$

**Standard translation**

$\mathrm{st}_x : ML \to FOL$ s.t. $\mathrm{st}_x(\varphi) \in FOL$ has one free variable $x$

**Goal**

$\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$

**Proposition letters**

$\mathrm{st}_x(p) := Px$

$$\mathfrak{M}, w \Vdash p \iff w \in V(p)$$
$$\iff w \in P_{\mathfrak{M}} \quad \iff \mathfrak{M} \models Px[w]$$

# The standard translation

(BRV: §2.4)

Definition

Recursively define $\mathrm{st}_x : ML \to FOL$ by

$\mathrm{st}_x(p) := Px$

$\mathrm{st}_x(\top) := (x = x)$

$\mathrm{st}_x(\neg\varphi) := \neg\,\mathrm{st}_x(\varphi)$

$\mathrm{st}_x(\varphi \wedge \psi) := \mathrm{st}_x(\varphi) \wedge \mathrm{st}_x(\psi)$

$\mathrm{st}_x(\Box\varphi) := \forall y(xRy \to \mathrm{st}_y(\varphi))$

Theorem

$\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$

Proof

By induction. Base cases okay.

$\neg\varphi$ $\quad \mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi \overset{IH}{\iff} \mathfrak{M} \not\models \mathrm{st}_x(\varphi)[w] \iff \mathfrak{M} \models \neg\,\mathrm{st}_x(\varphi)[w]$

# The standard translation

(BRV: §2.4)

**Theorem** $\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$

**Proof** By induction. Base cases okay.

$\neg\varphi$ $\quad \mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi \overset{IH}{\iff} \mathfrak{M} \not\models \mathrm{st}_x(\varphi)[w] \iff \mathfrak{M} \models \neg\,\mathrm{st}_x(\varphi)[w]$

**Exercise 7** Prove the case for $\Box\varphi$

- Assume $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$ for all $x, w$
- Prove $\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\Box\varphi)[w]$

# The standard translation

(BRV: §2.4)

**Exercise 7**

Prove the case for $\Box\varphi$

- Assume $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$ for all $x, w$
- Prove $\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\Box\varphi)[w]$

**Proof**

We have:

$$\begin{aligned} \mathfrak{M}, w \Vdash \Box\varphi &\iff \forall v \in W(wRv \to \mathfrak{M}, v \Vdash \varphi) \\ &\iff \forall v \in W(wRv \to \mathrm{st}_x(\varphi)[v]) \\ &\iff \forall v \in W(wRv \to \mathrm{st}_v(\varphi)) \\ &\iff \mathrm{st}_x(\Box\varphi)[w] \end{aligned}$$

# Tense modal logic

Recall that tense logic $TL$ extends $ML$ with operators $\blacksquare$ (and $\blacklozenge$)

Exercise 8

(a) Extend $\mathrm{st}_x$ to a map $\mathrm{st}_x : TL \to FOL$

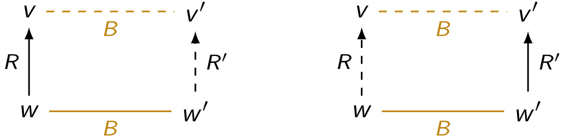
(b) Prove that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models \mathrm{st}_x(\varphi)[w]$ for all $\varphi \in TL$

# Bisimulations

# Bisimulations

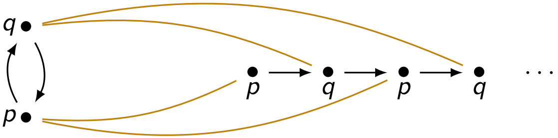
(BRV: §2.2)

A bisimulation between $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ is $B \subseteq W \times W'$ s.t.

- $w \in V(p)$ iff $w' \in V'(p)$, for all $(w, w') \in B$
- 


Write $w \rightleftharpoons w'$ if $wBw'$ for some bisimulation

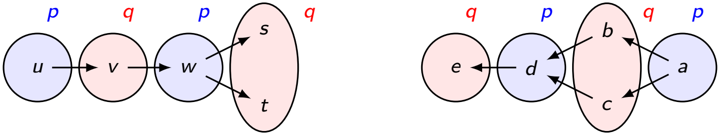
Example



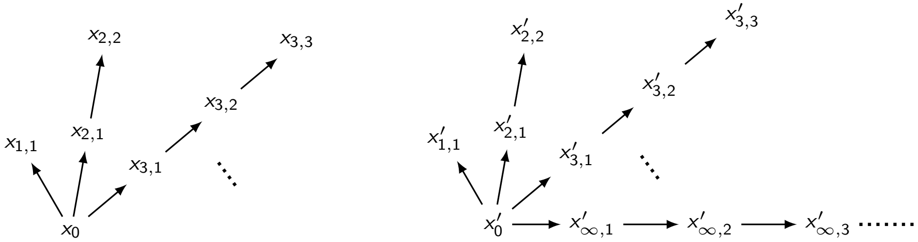
# Bisimulations

(BRV: §2.2)

**Exercise 9** Find a bisimulation between the following two models:




**Exercise 10** Prove that $x_0$ and $x_0'$ are not bisimilar:



# Some properties of bisimulations

A bisimulation between $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ is $B \subseteq W \times W'$ s.t.

- $w \in V(p)$ iff $w' \in V'(p)$, for all $(w, w') \in B$
- If $wBw'$ and $wRv$ then $\exists v' \in W'$ s.t. $w'R'v'$ and $vBv'$
- If $wBw'$ and $w'R'v'$ then $\exists v \in W$ s.t. $wRv$ and $vBv'$

Exercise 11 Let $B$, $D$ be bisimulations between $\mathfrak{M}$ and $\mathfrak{M}'$, and $S$ a bisimulation between $\mathfrak{M}'$ and $\mathfrak{M}''$. Show that the following are bisimulations as well:

(a) $B \cup D$ (and what about $B \cap D$?)

(b) $B^{-1} = \{(w', w) \mid (w, w') \in B\}$

(c) $B \circ S = \{(w, w'') \mid \exists x' \text{ s.t. } (w, x') \in B \text{ and } (x', w'') \in S\}$

(d) $id_{\mathfrak{M}} = \{(w, w) \mid w \in \mathfrak{M}\}$

# The Hennessy-Milner property

# Adequacy

(BRV: §2.2)

**Theorem** If $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$ then $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$

**Proof** We prove $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}', w' \Vdash \varphi$ by induction on $\varphi$

$\varphi = p$ By definition

$\varphi = \neg\varphi'$ $\mathfrak{M}, w \Vdash \neg\varphi'$ iff $\mathfrak{M}, w \nVdash \varphi'$ iff (IH) $\mathfrak{M}', w' \nVdash \varphi'$ iff $\mathfrak{M}', w' \Vdash \neg\varphi'$

$\varphi = \varphi_1 \wedge \varphi_2$ ...

$\varphi = \Box\varphi'$ Suppose $\mathfrak{M}, w \Vdash \Box\varphi'$. If $(w', v') \in R'$ then $v' \Vdash \varphi$ because


so $\mathfrak{M}', w' \Vdash \Box\varphi'$

# Temporal logic

(BRV: §2.2)

Exercise 12

(a) Adapt the definition of a bisimulation so it preserves $\blacksquare$ and $\blacklozenge$

(b) Prove that $\mathfrak{M}, w \rightleftharpoons_{\blacksquare\blacklozenge} \mathfrak{M}', w'$ implies $\mathfrak{M}, w \leftrightsquigarrow_{\blacksquare\blacklozenge} \mathfrak{M}', w'$

We define $\rightleftharpoons_{\blacksquare\blacklozenge}$ and $\leftrightsquigarrow_{\blacksquare\blacklozenge}$ as expected.

# The Hennessy-Milner property

(BRV: §2.2)

**Adequacy**

Bisimilarity implies modal equivalence:

$$\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w' \quad \text{implies} \quad \mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$$

**Definition**

A class K of models is called a Hennessy-Milner class if

$$\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w' \quad \text{if and only if} \quad \mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$$

**Theorem**

The image-finite models form a Hennessy-Milner class

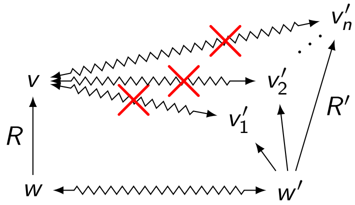
# Image-finite models

(BRV: §2.2)

**Theorem** The image-finite models form a Hennessy-Milner class

**Proof** Let $\mathfrak{M}$, $\mathfrak{M}'$ be image-finite models

**Claim** $\leftrightsquigarrow$ is a bisimulation

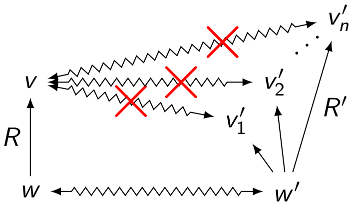



$$\mathfrak{M}, v \Vdash \varphi_1 \text{ and } \mathfrak{M}', v'_1 \nVdash \varphi_1$$
$$\mathfrak{M}, v \Vdash \varphi_2 \text{ and } \mathfrak{M}', v'_2 \nVdash \varphi_2$$
$$\vdots \qquad \vdots$$
$$\mathfrak{M}, v \Vdash \varphi_n \text{ and } \mathfrak{M}', v'_n \nVdash \varphi_n$$

But then: $\mathfrak{M}, w \Vdash \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$ and $\mathfrak{M}', w' \nVdash \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$ ↯

# Modally saturated models

(BRV: §2.5)




$\mathfrak{M}, v \Vdash \varphi_1$ and $\mathfrak{M}', v'_1 \nVdash \varphi_1$

$\mathfrak{M}, v \Vdash \varphi_2$ and $\mathfrak{M}', v'_2 \nVdash \varphi_2$

$\vdots \qquad \vdots$

$\mathfrak{M}, v \Vdash \varphi_n$ and $\mathfrak{M}', v'_n \nVdash \varphi_n$

**Definition** A model $\mathfrak{M}' = (W', R', V')$ is m-saturated if for all $w' \in W'$:

If $\Sigma \subseteq ML$ and every finite $\Sigma' \subseteq \Sigma$ is satisfied at some $v' \in R[w']$,
then $\Sigma$ is satisfied at some $v' \in R[w']$.

**Theorem** The m-saturated models form a Hennessy-Milner class

# The Hennessy-Milner property

**Exercise 13** Prove that the m-saturated models form a Hennessy-Milner class

**Exercise 14** Give a model that is modally saturated but not image-finite

**Exercise 15** Prove that for m-saturated models $\mathfrak{M}$ and $\mathfrak{M}'$:

$$\mathfrak{M}, w \leftrightsquigarrow_{\blacksquare\blacklozenge} \mathfrak{M}', w' \quad \text{implies} \quad \mathfrak{M}, w \rightleftharpoons_{\blacksquare\blacklozenge} \mathfrak{M}', w'$$

# Omega-saturated models

# $\omega$-saturation

(BRV: §2.6)

**Definition** Fix a Kripke model $\mathfrak{M} = (W, R, V)$ and $A \subseteq W$

- Let $FOL[A]$ be the extension of $FOL$ with constants $\{\underline{a} \mid a \in A\}$
- The model $\mathfrak{M}_A$ extends $\mathfrak{M}$ with $I(\underline{a}) = a \in W$

**Definition** $\mathfrak{M}$ is $\omega$-saturated if for every finite $A \subseteq W$ and all $\Gamma(x) \subseteq FOL[A]$: If $\mathfrak{M}_A \models \Delta$ for all finite $\Delta \subseteq \Gamma(x)$, then $\mathfrak{M}_A \models \Gamma(x)$

**Proposition** Every $\omega$-saturated model is modally saturated

# $\omega$-saturation versus modal saturation

(BRV: §2.6)

**Definition**

Fix a Kripke model $\mathfrak{M} = (W, R, V)$ and $A \subseteq W$

- Let $FOL[A]$ be the extension of $FOL$ with constants $\{\underline{a} \mid a \in A\}$
- The model $\mathfrak{M}_A$ extends $\mathfrak{M}$ with $I(\underline{a}) = a \in W$

**Definition**

$\mathfrak{M}$ is $\omega$-saturated if for every finite $A \subseteq W$ and all $\Gamma(x) \subseteq FOL[A]$: If $\mathfrak{M}_A \models \Delta$ for all finite $\Delta \subseteq \Gamma(x)$, then $\mathfrak{M}_A \models \Gamma(x)$

**Proposition**

Every $\omega$-saturated model $\mathfrak{M}$ is modally saturated

**Proof**

Suppose $w \in W$ and $\Sigma \subseteq ML$ is finitely satisfiable in $R[w]$.
Take $A = \{w\}$ and let $\Gamma(x) = \{R\underline{w}x\} \cup \{\mathrm{st}_x(\varphi) \mid \varphi \in \Sigma\}$.
Then $\mathfrak{M}_A \models \Delta$ for all finite $\Delta \subseteq \Gamma(x)$, so $\mathfrak{M}_A \models \Gamma(x)$.
It follows that $\Sigma$ is satisfiable in $R[w]$.

# Overview

Yesterday

- Language and semantics of modal logic ✓
- Other modal operators ✓
- Correspondence ✓

Today

- Translation into first-order logic ✓
- Bisimulations ✓
- Hennessy-Milner theorems ✓

Today

- Constructing $\omega$-saturated models
- Van Benthem characterisation theorem
- Variations