Modal Logic

ANU Logic Summer School

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Overview

Wednesday

- Language and semantics of modal logic ✓
- Other modal operators ✓
- Correspondence ✓

Yesterday

- Translation into first-order logic ✓
- Bisimulations ✓
- Hennessy-Milner theorems ✓

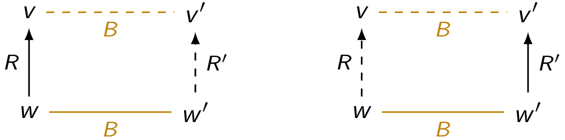
Today

- Constructing ω-saturated models
- Van Benthem characterisation theorem
- Variations

Bisimulations

(BRV: §2.2 and §2.5)

A bisimulation between $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ is $B \subseteq W \times W'$ s.t.

- $w \in V(p)$ iff $w' \in V'(p)$, for all $(w, w') \in B$
- 

Adequacy $\quad \mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$ implies $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$

HM theorem $\quad$ For the class of modally saturated models:

$$\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w' \quad \text{if and only if} \quad \mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$$

Making models omega-saturated

ω-saturation

(BRV: §2.6)

Definition Fix a Kripke model $\mathfrak{M} = (W, R, V)$ and $A \subseteq W$

- Let $FOL[A]$ be the extension of FOL with constants $\{\underline{a} \mid a \in A\}$
- The model $\mathfrak{M}_A$ extends $\mathfrak{M}$ with $I(\underline{a}) = a \in W$

Definition $\mathfrak{M}$ is ω-saturated if for every finite $A \subseteq W$ and all $\Gamma(x) \subseteq FOL[A]$: If $\mathfrak{M}_A \models \Delta$ for all finite $\Delta \subseteq \Gamma(x)$, then $\mathfrak{M}_A \models \Gamma(x)$

Proposition Every ω-saturated model is modally saturated

Ultrafilters

(BRV: §2.6)

Definition An ultrafilter over a set I is a nonempty subset $U \subseteq \mathcal{P}I$ such that

- $I \in U$
- If $a, b \in U$ then $a \cap b \in U$
- For all $a \in \mathcal{P}I$, either $a \in U$ or $I \setminus a \in U$

Definition An ultrafilter U on a set I is called countably incomplete if it is not closed under countable intersections

Ultraproducts of sets

(BRV: §2.6)

U-equivalence

Let $\{W_i \mid i \in I\}$ be some I-indexed collection of sets.

- Their product $\prod_{i\in I} W_i$ consists functions f with domain I s.t. $f(i) \in W_i$
- Two such functions f and g are U-equivalent if

$$\{i \in I \mid f(i) = g(i)\} \in U$$

- This gives an equivalence relation $\sim_U$ on the product of the W_i, we write f_U for the equivalence class of f

Ultraproduct

The ultraproduct of sets W_i (indexed by I) is

$$\prod_U W_i = \{f_U \mid f \in \prod_{i\in I} W_i\}$$

Ultraproducts of models

(BRV: §2.6)

Definition The ultraproduct $\prod_U \mathfrak{M}_i$ for models $\mathfrak{M}_i = (W_i, R_i, V_i)$ is the model $\mathfrak{M}_U := (W_U, R_U, V_U)$ where

- $W_U = \prod_U W_i$
- $f_U R_U g_U$ iff $\{i \in I \mid f(i) R g(i)\} \in U$
- $f_U \in V_U(p)$ iff $\{i \in I \mid f(i) \in V_i(p)\} \in U$

Proposition Let $\prod_U \mathfrak{M}$ be an ultrapower of $\mathfrak{M}$, and let f_w be defined by $f_w(i) = w$ for all $i \in I$. Then

$$\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{M}_U, (f_w)_U \Vdash \varphi$$

Ultraproducts of models

(BRV: §2.6)

Definition

The ultraproduct $\prod_U \mathfrak{M}_i$ for models $\mathfrak{M}_i = (W_i, R_i, V_i)$ is the model $\mathfrak{M}_U := (W_U, R_U, V_U)$ where

- $W_U = \prod_U W_i$
- $f_U R_U g_U$ iff $\{i \in I \mid f(i) R g(i)\} \in U$
- $f_U \in V_U(p)$ iff $\{i \in I \mid f(i) \in V_i(p)\} \in U$

Proposition

Let $\prod_U \mathfrak{M}$ be an ultrapower of $\mathfrak{M}$, and let f_w be defined by $f_w(i) = w$ for all $i \in I$. Then for all $\alpha(x) \in FOL$,

$$\mathfrak{M} \models \alpha(x)[w] \quad \text{iff} \quad \mathfrak{M}_U \models \alpha(x)[(f_w)_U]$$

Proposition

Let U be a countably incomplete ultrafilter over a non-empty set I, and $\mathfrak{M}$ a Kripke model. Then $\prod_U \mathfrak{M}$ is omega-saturated.

Ultra takeaway

(BRV: §2.6)

For each Kripke model $\mathfrak{M}$ we can construct some model $\mathfrak{M}^*$ such that

- $\mathfrak{M}^*$ is ω-saturated
- there exists an injective map

$$f : \mathfrak{M} \to \mathfrak{M}^* : w \mapsto w^*$$

that preserves truth of formulas

Bisimilarity-somewhere-else

Theorem $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v$ if and only if $\mathfrak{M}^*, w^* \rightleftharpoons \mathfrak{N}^*, v^*$

Proof

elem. ext.

H-M theorem

$$\mathfrak{M}^*, w^* \leftrightsquigarrow \mathfrak{N}^*, v^*$$

Van Benthem characterisation theorem

Van Benthem characterisation theorem

(BRV: §2.6)

Definition A FOL-formulae $\alpha(x)$ is *invariant under bisimulations* if for every bisimulation B between $\mathfrak{M}$ and $\mathfrak{M}'$, $(w, w') \in B$ implies

$$\mathfrak{M} \models \alpha(x)[w] \quad \text{iff} \quad \mathfrak{M}' \models \alpha(x)[w']$$

VB theorem Let $\alpha(x)$ be a FOL-formula with one free variable x. TFAE:

- $\alpha(x)$ is equivalent to $\mathrm{st}_x(\varphi)$ for some $\varphi \in ML$
- $\alpha(x)$ is invariant under bisimulations

Proof $(\Downarrow)$ Suppose $\alpha(x) = \mathrm{st}_x(\varphi)$ and $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$, then:

$$\mathfrak{M} \models \mathrm{st}_x(\varphi)[w] \iff \mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}', w' \Vdash \varphi \iff \mathfrak{M}' \models \mathrm{st}_x(\varphi)[w']$$

Van Benthem characterisation theorem

(BRV: §2.6)

VB theorem $\alpha(x) \equiv \mathrm{st}_x(\varphi)$ iff $\alpha(x)$ is invariant under bisim

Proof $(\Uparrow)$ Suffices: $\underbrace{\{\mathrm{st}_x(\varphi) \mid \varphi \in ML \text{ and } \alpha(x) \models \mathrm{st}_x(\varphi)\}}_{MOC(\alpha)} \models \alpha(x)$

Then $\bigwedge X \models \alpha(x)$ for some finite $X \subseteq MOC(\alpha)$

By construction $\alpha(x) \models \bigwedge X$

$$\begin{aligned} \text{So } \alpha(x) \equiv \bigwedge X &= \bigwedge\{\mathrm{st}_x(\varphi_1), \dots, \mathrm{st}_x(\varphi_n)\} \\ &= \mathrm{st}_x(\varphi_1) \wedge \dots \wedge \mathrm{st}_x(\varphi_n) \\ &= \mathrm{st}_x(\varphi_1 \wedge \dots \wedge \varphi_n) \end{aligned}$$

Van Benthem characterisation theorem

(BRV: §2.6)

VB theorem $\alpha(x) \equiv \mathrm{st}_x(\varphi)$ iff $\alpha(x)$ is invariant under bisim

Proof ($\Uparrow$) Suffices: $\underbrace{\{\mathrm{st}_x(\varphi) \mid \varphi \in ML \text{ and } \alpha(x) \models \mathrm{st}_x(\varphi)\}}_{MOC(\alpha)} \models \alpha(x)$ ✓

assume: $\mathfrak{M} \models MOC(\alpha)[w]$ need: $\mathfrak{M} \models \alpha(x)[w]$

claim: $\underbrace{\{\mathrm{st}_x(\varphi) \mid \mathfrak{M} \models \mathrm{st}_x(\varphi)[w]\}}_{T(x)} \cup \{\alpha(x)\}$ is consistent

If not, then by compactness there exists a finite set $Y \subseteq T(x)$ s.t. $Y \cup \{\alpha(x)\}$ is inconsistent, i.e. $\vdash\models \neg(\alpha(x) \wedge \bigwedge Y)$

$$\neg(\alpha(x) \wedge \bigwedge Y) \equiv \neg\alpha(x) \vee \neg(\bigwedge Y) \equiv \alpha(x) \to \neg \bigwedge Y$$

Van Benthem characterisation theorem

(BRV: §2.6)

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If not, then by compactness there exists a finite set $Y \subseteq T(x)$ s.t. $Y \cup \{\alpha(x)\}$ is inconsistent, i.e. $\models \alpha(x) \to \neg(\bigwedge Y)$.

Then $\alpha(x) \models \neg(\bigwedge Y)$ for some $Y = \{\mathrm{st}_x(\varphi_1), \dots, \mathrm{st}_x(\varphi_n)\}$,
so $\alpha(x) \models \mathrm{st}_x(\neg(\varphi_1 \wedge \dots \wedge \varphi_n))$ ↯

Van Benthem characterisation theorem (BRV: §2.6)

VB theorem $\alpha(x) \equiv \mathrm{st}_x(\varphi)$ iff $\alpha(x)$ is invariant under bisim

Proof ($\Uparrow$) Suffices: $\underbrace{\{\mathrm{st}_x(\varphi) \mid \varphi \in ML \text{ and } \alpha(x) \models \mathrm{st}_x(\varphi)\}}_{MOC(\alpha)} \models \alpha(x)$ ✓

assume: $\mathfrak{M} \models MOC(\alpha)[w]$ need: $\mathfrak{M} \models \alpha(x)[w]$

claim: $\underbrace{\{\mathrm{st}_x(\varphi) \mid \mathfrak{M} \models \mathrm{st}_x(\varphi)[w]\}}_{T(x)} \cup \{\alpha(x)\}$ is consistent ✓

Then $\mathfrak{N} \models T(x) \cup \{\alpha(x)\}[v]$ for some $\mathfrak{N}, v$

$$\begin{array}{ccc} \mathfrak{M}, w & \leftrightsquigarrow & \mathfrak{N}, v \\ \downarrow & & \downarrow \\ \mathfrak{M}^*, w^* & \rightleftharpoons & \mathfrak{N}^*, v^* \end{array}$$

$\mathfrak{N} \models \alpha(x)[v] \Rightarrow \mathfrak{N}^* \models \alpha(x)[v^*]$

$\Rightarrow \mathfrak{M}^* \models \alpha(x)[w^*]$

$\Rightarrow \mathfrak{M} \models \alpha(x)[w]$

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- Hennessy-Milner theorems ✓

Today

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- Van Benthem characterisation theorem ✓
- Variations

Variations

- Tense modal logic
- Positive modal logic (Theorem 3.5 in this paper)
- Intuitionistic logic (Theorem 5.2 in the same paper)
- Instantial neighbourhood logic (Theorem 7.6 and 8.5 in this paper)
- And many more ...

Adaptation to temporal logic

Definition: A FOL-formulae $\alpha(x)$ is *invariant under tense bisimulations* if for every bisimulation B between $\mathfrak{M}$ and $\mathfrak{M}'$, $(w, w') \in B$ implies

$$\mathfrak{M} \models \alpha(x)[w] \quad \text{iff} \quad \mathfrak{M}' \models \alpha(x)[w']$$

Theorem: Let $\alpha(x)$ be a FOL-formula with one free variable x. TFAE:

- $\alpha(x)$ is equivalent to $\mathrm{st}(\varphi)$ for some $\varphi \in TL$
- $\alpha(x)$ is invariant under tense bisimulations

Exercise 16 Prove this.

Van Benthem for positive modal logic

Positive modal logic

Language *PML* $\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \qquad p \in \mathsf{Prop}$

Interpretation . . . in Kripke models, the same as for *ML*

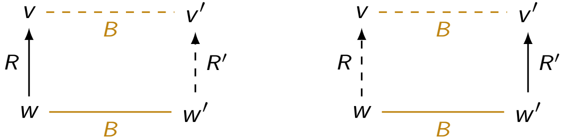
Standard translation Recursively define $\mathrm{st}_x : PML \to FOL$ by . . .

$$\mathrm{st}_x(\Box\varphi) := \forall y(xRy \to \mathrm{st}_y(\varphi))$$
$$\mathrm{st}_x(\Diamond\varphi) := \exists y(xRy \wedge \mathrm{st}_y(\varphi))$$

Theorem $\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{M} \models \mathrm{st}_x(\varphi)[w] \qquad \text{for all } \varphi \in PML$

Simulations

A simulation from $\mathfrak{M} = (W, R, V)$ to $\mathfrak{M}' = (W', R', V')$ is a relation $S \subseteq W \times W'$ such that

- if $w \in V(p)$ then $w' \in V'(p)$, for all $(w, w') \in B$
- 

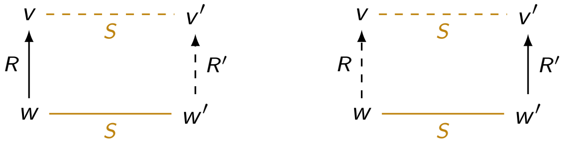
Write $w \rightharpoonup w'$ if there exists a simulation S such that wSw'.

Observations

- Every bisimulation is a simulation
- There exist simulations that are not bisimulations

Simulations

A simulation from $\mathfrak{M} = (W, R, V)$ to $\mathfrak{M}' = (W', R', V')$ is a relation $S \subseteq W \times W'$ such that

- if wBw' and $w \in V(p)$ then $w' \in V'(p)$
- 

Write $w \rightharpoonup w'$ if there exists a simulation S such that wSw'.

Theorem If $\mathfrak{M}, w \rightharpoonup \mathfrak{M}', w'$ then for all $\varphi \in PML$:

$$\mathfrak{M}, w \Vdash \varphi \quad \text{implies} \quad \mathfrak{M}', w' \Vdash \varphi$$

Exercise num Prove this

Hennessy-Milner for simulations

Modal inclusion

If $\mathfrak{M}, w \Vdash \varphi$ implies $\mathfrak{M}', w' \Vdash \varphi$, for all $\varphi \in PML$ then we write

$$\mathfrak{M}, w \rightsquigarrow \mathfrak{M}', w'$$

Positive HM classes

A class K of Kripke model is a positive Hennessy-Milner class if

$$\mathfrak{M}, w \rightharpoonup \mathfrak{M}', w' \quad \text{if and only if} \quad \mathfrak{M}, w \rightsquigarrow \mathfrak{M}', w'$$

Theorem

The class of image-finite models is a positive Hennessy-Milner class

Exercise num

Prove this theorem

Van Benthem for positive modal logic

Definition A FOL-formulae $\alpha(x)$ is preserved by simulations if for every simulation S from $\mathfrak{M}$ to $\mathfrak{M}'$, $(w, w') \in S$ implies

$$\mathfrak{M} \models \alpha(x)[w] \quad \text{implies} \quad \mathfrak{M}' \models \alpha(x)[w']$$

Theorem Let $\alpha(x)$ be a FOL-formula with one free variable x. TFAE:

- $\alpha(x)$ is equivalent to $\mathrm{st}(\varphi)$ for some $\varphi \in ML^+$
- $\alpha(x)$ is preserved by simulations

Exercise 17 prove this :-)

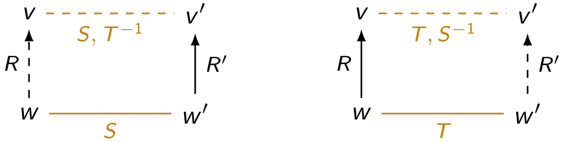
Van Benthem for intuitionistic logic

Positive logic + strict implication

Language $PL_{\strictif}$	$\varphi ::= p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \strictif \varphi \qquad p \in \text{Prop}$
Interpretation ...	in Kripke models, the same as for ML
Observation	Restricting to reflexive transitive Kripke models gives intuitionistic logic
Standard translation	Recursively define $\text{st}_x : PL_{\strictif} \to FOL$ by ...
	$\text{st}_x(\varphi \strictif \psi) := \forall y((xRy \wedge \text{st}_y(\varphi)) \to \text{st}_y(\psi))$
Theorem	$\mathfrak{M}, w \Vdash \varphi \quad \text{iff} \quad \mathfrak{M} \models \text{st}_x(\varphi)[w] \qquad \text{for all } \varphi \in PL_{\strictif}$

Intuitionistic simulations

An intuitionistic bisimulation from $\mathfrak{M} = (W, R, V)$ to $\mathfrak{M}' = (W', R', V')$ is a pair of relations $S \subseteq W \times W'$ and $T \subseteq W' \times W$ such that

- if wSw' and $w \in V(p)$ then $w' \in V'(p)$
- if $w'Tw$ and $w' \in V'(p)$ then $w \in V(p)$
- 

Write $w \rightleftharpoons_{\fishhook} w'$ if there exists a simulation S such that wSw'.

Theorem If $\mathfrak{M}, w \rightharpoonup \mathfrak{M}', w'$ and $\mathfrak{M}, w \Vdash \varphi$ then $\mathfrak{M}', w' \Vdash \varphi$, for all $\varphi \in PL_{\fishhook}$

Van Benthem for $PL_{\to\exists}$

Definition

A FOL-formulae $\alpha(x)$ is preserved by simulations if for every intuitionistic simulation (S, T) from $\mathfrak{M}$ to $\mathfrak{M}'$, $(w, w') \in S$ implies

$$\mathfrak{M} \models \alpha(x)[w] \quad \text{implies} \quad \mathfrak{M}' \models \alpha(x)[w']$$

Theorem

Let $\alpha(x)$ be a FOL-formula with one free variable x. TFAE:

- $\alpha(x)$ is equivalent to $\mathrm{st}(\varphi)$ for some $\varphi \in PL_{\to\exists}$
- $\alpha(x)$ is preserved by intuitionistic bisimulations

Theorem

Let $\alpha(x)$ be a FOL-formula with one free variable x. TFAE:

- $\alpha(x)$ is equivalent over preordered models to $\mathrm{st}(\varphi)$ for some $\varphi \in PL_{\to\exists}$
- $\alpha(x)$ is preserved by intuitionistic bisimulations

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