Gödel's Theorem Without Tears

Synthetic Computability and Mechanised Incompleteness

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Gödel's Theorem Without Tears?

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CHURCH'S THESIS WITHOUT TEARS

FRED RICHMAN

§1. **Introduction.** The modern theory of computability is based on the works of Church, Markov and Turing who, starting from quite different models of computation, arrived at the same class of computable functions. The purpose of this paper is the show how the main results of the Church-Markov-Turing theory of computable functions may quickly be derived and understood without recourse to the largely irrelevant theories of recursive functions, Markov algorithms, or Turing machines. We do this by ignoring the problem of what constitutes a computable function and concentrating on the central feature of the Church-Markov-Turing theory: that the set of computable partial functions can be effectively enumerated. In this manner we are led directly to the heart of the theory of computability without having to fuss about what a computable function is.

Get to the heart of computational incompleteness proofs
without having to fuss about what a computable function is!

Gödel's Theorem Without Tears!

Gödel's Theorem Without Tears

Essential Incompleteness in Synthetic Computability

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Abstract

Gödel published his groundbreaking first incompleteness theorem in 1931, stating that a large class of formal logics admits independent sentences which are neither provable nor refutable. This result, in conjunction with his second incompleteness theorem, established the impossibility of concluding Hilbert's program, which pursued a possible path towards a single formal system unifying all of mathematics. Using a technical trick to refine Gödel's original proof, the incompleteness result was strengthened further by Rosser in 1936 regarding the conditions imposed on the formal systems.

`https://www.ps.uni-saarland.de/extras/incompleteness/`

Course Structure

I'll walk you through the paper contents, explaining underlying concepts:

1. Basic Synthetic Computability Theory
2. Weak Gödel à la Turing
3. More Basic Synthetic Computability Theory
4. Strong Gödel à la Kleene
5. Instantiation to FOL

Course Topics

We'll touch a lot of areas, slow me down any time!

- Computability Theory
  - Decidability, semi-decidability, recursive inseparability...
- Metalogic
  - First-order logic, soundness/consistency, incompleteness...
- Constructive Mathematics
  - Non-constructive axioms, anti-classical axioms, synthetic computability...
- Computer Mechanisation
  - Coq proof assistant, dependent type theory...

The First Incompleteness Theorem

Which formal systems $\mathcal{S}$ admit sentences φ with both $\mathcal{S} \nvdash \varphi$ and $\mathcal{S} \nvdash \neg\varphi$?

- Gödel: all sound, sufficiently expressive ones (Gödel, 1931)
- Rosser: all consistent, sufficiently expressive ones (Rosser, 1936)
- Turing(/Post): Gödel's incompleteness follows from undecidability
- Kleene: Rosser's incompleteness follows from recursive inseparability (Kleene, 1951)

Motivational Testimonies

Computational proofs of Rosser's strength are not well-known but desirable:

- *"Recently I was struck to discover just such a proof laid out..."*
  Anatoly Vorobey on the FOM mailing list
- *"A few months ago, I found a short, simple, Turing-machine-based proof of Rosser's Theorem... So, will Gödel's Theorem always and forevermore be taught as a centerpiece of computability theory, and will the Gödel numbers get their much-deserved retirement?"*
  Scott Aaronson on his blog
- *"Here I shall present very simple computability-based proofs of Gödel/Rosser's incompleteness theorem, which require only basic knowledge about programs. I feel that these proofs are little known despite giving a very general form of the incompleteness theorems, and also easy to make rigorous without even depending on much background knowledge in logic."*
  User21820 on StackExchange

Motivational Testimonies (ctd.)

Gödel's incompleteness theorems [edit]

The concepts raised by Gödel's incompleteness theorems are very similar to those raised by the halting problem, and the proofs are quite similar. In fact, a weaker form of the First Incompleteness Theorem is an easy consequence of the undecidability of the halting problem. This weaker form differs from the standard statement of the incompleteness theorem by asserting that an axiomatization of the natural numbers that is both complete and sound is impossible. The "sound" part is the weakening: it means that we require the axiomatic system in question to prove only *true* statements about natural numbers. Since soundness implies consistency, this weaker form can be seen as a corollary of the strong form. It is important to observe that the statement of the standard form of Gödel's First Incompleteness Theorem is completely unconcerned with the truth value of a statement, but only concerns the issue of whether it is possible to find it through a mathematical proof.

https://en.wikipedia.org/wiki/Halting_problem

The Matrix of Incompleteness Statements

	Unexpected decidability	Disprove completeness	Independent sentence
Soundness	(Tue)	Turing (Wed)	Gödel (Thu)
ω-consistency			Gödel
Consistency			Rosser/Kleene (Fri)

The proofs given in this course will be:

- Based on computability theory
- In an abstract and synthetic setting
- Mechanised in Coq (runs in browser!)

Part 1

Basic Synthetic Computability Theory

There are non-computable functions!?

Standard example: the characteristic function of the halting problem

$$\chi_K(M) \ := \begin{cases} 1 & \text{if } M \text{ terminates} \\ 0 & \text{if } M \text{ diverges} \end{cases}$$

Three properties ensure that this relation is a function:

- Functionality: obvious
- Totality: given M, since M either terminates or diverges by the law of excluded middle, we have either $\chi_K(M) = 1$ or $\chi_K(M) = 0$, respectively
- Unique choice: total functional relations are actually functions

So classical logic is needed to show that $\chi_K(M)$ really is a function!

Pros and Cons of the Excluded Middle

Classical reasoning strengthens logical system:

- Enables proofs by contradiction, classical case analysis, contraposition
- Constructive proofs, if possible, can be considerably more complicated
- Embraces Platonic perspective on mathematics

Constructive logic unveils constructive content:

- Every definable function is computable
- Every proof of existential (disjunctive) statements contains witness (decision)
- Admits a more agnostic perspective on mathematics

Some Synthetic Definitions

$P \subseteq \mathbb{N}$ is decidable	if there exists $d : \mathbb{N} \to \mathbb{B}$	with $x \in P \leftrightarrow d\, x = \mathrm{tt}$
$P \subseteq \mathbb{N}$ is semi-decidable	if there exists $s : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$	with $x \in P \leftrightarrow \exists n.\, s\,(x, n) = \mathrm{tt}$
$P \subseteq \mathbb{N}$ is enumerable	if there exists $e : \mathbb{N} \to \mathbb{N} \cup \{*\}$	with $x \in P \leftrightarrow \exists n.\, e\, n = x$
$P \subseteq \mathbb{N}$ reduces to $Q \subseteq \mathbb{N}$	if there exists $r : \mathbb{N} \to \mathbb{N}$	with $x \in P \leftrightarrow r\, x \in Q$

Some Synthetic Properties

1. Decidable sets are semi-decidable and co-semi-decidable
   $\Rightarrow$ Given $d : \mathbb{N} \to \mathbb{B}$ pick $s(x, n) := d\,x$ or $s(x, n) := \neg_{\mathbb{B}} d\,x$, respectively

2. A set is semi-decidable if and only if it is enumerable
   $\Rightarrow$ Given a semi-decider $s : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$ pick the enumerator

$$e\,n := \begin{cases} n_2 & n = \langle n_1, n_2 \rangle \text{ and } s(n_1, n_2) = \mathrm{tt} \\ * & \text{otherwise} \end{cases}$$

   and given an enumerator $e : \mathbb{N} \to \mathbb{N} \cup \{*\}$ pick the semi-decider $s(x, n) := e\,n \stackrel{?}{=} x$

3. If X reduces to Y and Y is decidable, then so is X
   $\Rightarrow$ Given $d : \mathbb{N} \to \mathbb{B}$ and $f : \mathbb{N} \to \mathbb{N}$ pick $d \circ f$

Generalised Synthetic Definitions

Synthetic definitions immediately transport to arbitrary sets:

$P \subseteq X$ is decidable	if there exists $d : X \to \mathbb{B}$	with $x \in P \leftrightarrow d\,x = \mathrm{tt}$
$P \subseteq X$ is semi-decidable	if there exists $s : X \times \mathbb{N} \to \mathbb{B}$	with $x \in P \leftrightarrow \exists n.\, s\,(x, n) = \mathrm{tt}$
$P \subseteq X$ is enumerable	if there exists $e : \mathbb{N} \to X \,\dot{\cup}\, \{*\}$	with $x \in P \leftrightarrow \exists n.\, e\,n = x$
$P \subseteq X$ reduces to $Q \subseteq Y$	if there exists $r : X \to Y$	with $x \in P \leftrightarrow r\,x \in Q$

Post's Theorem and Markov's Principle

If P is semi-decidable and co-semi-decidable, it should be decidable!

- Idea: run both semi-deciders in parallel, pick the one terminating
- Problem: we only know that this procedure will not diverge but need actual termination
- Solution: assume Markov's Principle, a very restricted form of double-negation elimination

$$\forall f : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\exists n.\, f\ n = \mathrm{tt}) \to \exists n.\, f\ n = \mathrm{tt}$$

Theorem

Assuming Markov's Principle, if P is semi-decidable and co-semi-decidable, then it is decidable. In fact, the converse also holds, so the assumption of Markov's Principle is necessary.

Post's Theorem and Markov's Principle (Proof)

Theorem

Assuming Markov's Principle, if P is semi-decidable and co-semi-decidable, then it is decidable. In fact, the converse also holds (for non-empty P), so Markov's Principle is necessary.

Proof.

Given s_1 semi-deciding P and s_2 semi-deciding $\overline{P}$, we can show:

$$\forall x.\, \exists n.\, s_1\,(x, n) \vee_{\mathbb{B}} s_2\,(x, n) = \mathrm{tt} \qquad (*)$$

1. To show $(*)$, assume x and apply Markov's Principle, so it suffices to assume $\neg(\exists n.\, s_1\,(x, n) \vee_{\mathbb{B}} s_2\,(x, n) = \mathrm{tt})$ for a contradiction. Since $\neg\neg(x \in P \vee x \notin P)$ is provable, we can assume $x \in P \vee x \notin P$ which is enough to derive $\exists n.\, s_1\,(x, n) \vee_{\mathbb{B}} s_2\,(x, n) = \mathrm{tt}$.
2. Since $(*)$ induces a function $f : X \to \mathbb{N}$, we can construct $d\,x := s_1\,(x, f\,x)$.

For the converse, given $\neg\neg(\exists n.\, f\,n = \mathrm{tt})$ consider the set $P := \{x \in \mathbb{N} \mid \exists n.\, f\,n = \mathrm{tt}\}$. $\square$

Alternative Version of Post's Theorem

If we are in a setting that distinguishes computational versions of $\vee$ and $\exists$ from their truncated counterparts $|| \vee ||$ and $||\exists||$ with expectable properties, we can alternatively show:

Theorem

If P is bi-semi-decidable and satisfies $\forall x.\, ||x \in P \vee x \notin P||$, then it is decidable.

Proof.

Given s_1 semi-deciding P and s_2 semi-deciding $\overline{P}$, we can show:

$$\forall x.\, \exists n.\, s_1\,(x, n) \vee_{\mathbb{B}} s_2\,(x, n) = \mathrm{tt} \qquad (*)$$

1. To show $(*)$, assume x and apply linear search, so it suffices to show the truncated $||\exists n.\, s_1\,(x, n) \vee_{\mathbb{B}} s_2\,(x, n) = \mathrm{tt}||$. This then follows by case distinction on $||x \in P \vee x \notin P||$.
2. Since $(*)$ induces a function $f : X \to \mathbb{N}$, we can construct $d\,x := s_1\,(x, f\,x)$. □

Part 2
Weak Gödel à la Turing

Abstract Formal Systems

Definition

A triple $\mathcal{S} = (\mathbb{S}, \neg, \vdash)$ is called a formal system if:

- $\mathbb{S}$ is a set, considered the sentences of $\mathcal{S}$
- $\neg : \mathbb{S} \to \mathbb{S}$ is a function on sentences, considered the negation operation
- $\vdash \subseteq \mathbb{S}$ is a semi-decidable set of sentences, considered the provable sentences
- Consistency holds in the form that for all $\varphi : \mathbb{S}$ not both $\vdash \varphi$ and $\vdash \neg\varphi$

We call $\mathcal{S}$ complete if $\vdash \neg\varphi$ whenever $\nvdash \varphi$ for all $\varphi : \mathbb{S}$.

Instances:

- First-order axiomatisations like Q, PA, HA, ZF, IZF, ...
- Second-order arithmetics and set theories
- Simple and dependent type theories

Properties of Abstract Formal Systems

The following hold for any formal system $\mathcal{S} = (\mathbb{S}, \neg, \vdash)$:

1. The set $\{\varphi \in \mathbb{S} \mid \vdash \neg\varphi\}$ of refutable sentences is semi-decidable
   $\Rightarrow$ Given $s : \mathbb{S} \times \mathbb{N} \to \mathbb{B}$ semi-deciding the provable sentences, the function $s'(\varphi, n) := s(\neg\varphi, n)$ semi-decides the refutable sentences
2. Completeness implies that the unprovable and refutable sentences coincide
   $\Rightarrow$ Immediate, using consistency to show that refutable sentences are unprovable
3. Assuming Markov's Principle, completeness implies decidability of the provable sentences
   $\Rightarrow$ Combining (1) and (2) we obtain that the set of unprovable sentences is semi-decidable, then Post's theorem yields that the set of provable is decidable

Weak Gödel à la Turing

We say that a formal system $\mathcal{S}$ represents a set $P \subseteq X$ if P reduces to the provable sentences, i.e. if there is a function $r : X \to \mathbb{S}$ with

$$\forall x.\, x \in P \leftrightarrow \vdash r\, x.$$

Theorem

Assuming Markov's Principle, every set represented in a complete formal system is decidable. Put differently, every formal system that represents an undecidable set must be incomplete.

Unsatisfactory for at least (and a half) reasons:

- No independent sentence (not even an actual contradiction)
- Assumes Markov's Principle
- Representability is a soundness property

Part 3

More Basic Synthetic Computability Theory

Church's Thesis

To introduce actual undecidability, we need to assume an axiom restricting to the computational interpretation of functions!

Inconsistent attempt: assume a surjection $\mathbb{N} \to (\mathbb{N} \to \mathbb{B})$ to diagonalise against...

Consistent assumption in many variants of constructive mathematics:

- Kreisel (1970): "Every function can be captured by Kleene's T-predicate"
- Richman (1983): "The set of partial functions is countable"
- Bauer (2006): "The set of enumerable sets is enumerable"
- Swan and Uemura (2019): "Every function is computable by a Turing machine"
- Forster (2021a): "The set of partial functions is enumerable"

Partial Values

Definition

The set of partial values ∂X over a set X is defined by:

$$\partial X \;:=\; \{\xi : \mathbb{N} \to X \,\dot{\cup}\, \{*\} \mid \forall n n' x x'.\, \xi\, n = x \land \xi\, n' = x' \to x = x'\}$$

We write $\xi \downarrow x$ if $\xi\, n = x$ for some n, $\xi \downarrow$ if $\xi \downarrow x$ for some x, and $\xi \uparrow$ if there is no such x.

- If $\xi \downarrow x$ and $\xi \downarrow x'$ then $x = x'$
- There is a partial value $\xi_\perp$ with $\xi_\perp \uparrow$
- Given $x \in X$ there is a partial value ξ_x with $\xi_x \downarrow x$
- Given $\xi \in \partial X$ and $f : X \to \partial Y$ there is a partial value $\xi \gg= f$ with:

$$\xi \gg= f \downarrow y \;\leftrightarrow\; \exists x.\, \xi \downarrow x \land f\, x \downarrow y$$

Partial Functions

We write $X \rightharpoonup Y$ for $X \to \partial Y$:

1. Functions $X \to Y$ induce total partial functions $X \rightharpoonup Y$
   $\Rightarrow$ Given $f : X \to Y$ pick $g\,x := \xi_{fy}$
2. Total partial functions $X \rightharpoonup Y$ induce functions $X \to Y$
   $\Rightarrow$ A proof of $\forall x.\,\exists y.\,f\,x \downarrow y$ is a function $X \to Y$
3. Sets $P \subseteq X$ are semi-decidable iff they are the domain of a partial function
   $\Rightarrow$ Given $s : X \times \mathbb{N} \to \mathbb{B}$ pick $f : X \rightharpoonup \{\dagger\}$ defined by $f\,x\,n = \dagger$ iff $s\,(x, n) = \mathrm{tt}$.
4. Sets $P \subseteq X$ are enumerable iff they are the range of a partial function
   $\Rightarrow$ Given $e : \mathbb{N} \to X \,\dot\cup\, \{*\}$ pick $f : \mathbb{N} \rightharpoonup X$ defined by $f\,x\,n := e\,n$.

EPF and the Halting Problem

Axiom (EPF)

There is a universal function $\Theta : \mathbb{N} \to (\mathbb{N} \rightharpoonup \mathbb{B})$ *enumerating all partial functions:*

$$\forall f : \mathbb{N} \rightharpoonup \mathbb{B}.\, \exists c : \mathbb{N}.\, \forall xb.\, \Theta_c\, x \downarrow b \leftrightarrow f\, x \downarrow b$$

Lemma

The self-halting problem $\mathsf{K} := \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow\}$ *is semi-decidable but undecidable.*

Proof.

Assume $d : \mathbb{S} \to \mathbb{B}$ decides K. Consider the function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ with $f\, c \uparrow$ if $d\, c = \mathsf{tt}$ and $f\, c \downarrow \mathsf{tt}$ otherwise. Let c be the code of f given by EPF, we derive a contradiction:

$$d\, c = \mathsf{tt} \;\Leftrightarrow\; c \in K \;\Leftrightarrow\; \Theta_c\, c \downarrow \Leftrightarrow\; f\, c \downarrow \Leftrightarrow\; f\, c \downarrow \mathsf{tt} \;\Leftrightarrow\; d\, c = \mathsf{ff} \qquad \square$$

Stronger Gödel à la Turing

Since we now have an undecidable set, we immediately obtain:

Theorem

Assuming Markov's Principle, every formal system representing K *must be incomplete.*

Still unsatisfactory for three reasons:

- No independent sentence
- Assumes Markov's Principle
- Representability is a soundness property

Halting Problem (Refined)

Lemma

For every partial decider $d : \mathbb{N} \rightharpoonup \mathbb{B}$ *for* $\mathsf{K} = \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow\}$ *with*

$$\forall x.\, x \in \mathsf{K} \;\leftrightarrow\; d\, x \downarrow \mathsf{tt}$$

one can construct a concrete value c *such that* $d\, c$ *diverges.*

Proof.

We first define a partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ diagonalising against d by:

$$f\, x := \begin{cases} \xi_{\mathsf{tt}} & \text{if } d\, x \downarrow \mathsf{ff} \\ \xi_{\bot} & \text{otherwise} \end{cases}$$

Now using EPF we obtain a code c for f and deduce that $d\, c \uparrow$ by:

$$d\, c \downarrow \mathsf{tt} \;\Leftrightarrow\; c \in \mathsf{K} \;\Leftrightarrow\; \Theta_c\, c \downarrow \Leftrightarrow\; f\, c \downarrow \Leftrightarrow\; f\, c \downarrow \mathsf{tt} \;\Leftrightarrow\; d\, c \downarrow \mathsf{ff} \qquad \square$$

Post's Theorem (Refined)

Theorem

Given disjoint semi-decidable sets $P, Q \subseteq X$, there is a partial decider $d : X \rightharpoonup \mathbb{B}$ with:

$$\forall x.\, (x \in P \leftrightarrow d\,x \downarrow \mathrm{tt}) \land (x \in Q \leftrightarrow d\,x \downarrow \mathrm{ff})$$

Proof.

Given s_1 semi-deciding P and s_2 semi-deciding Q, define d by:

$$d\,x\,n := \begin{cases} \mathrm{tt} & \text{if } s_1\,x\,n \\ \mathrm{ff} & \text{if } s_2\,x\,n \\ * & \text{otherwise} \end{cases}$$

Then use disjointness to verify well-definedness and specification. □

Partial Deciders of Formal Systems

Since formal systems have two canonical semi-decidable sets:

Lemma

For every formal system $\mathcal{S} = (\mathbb{S}, \neg, \vdash)$ there is a partial function $d_\mathcal{S} : \mathbb{S} \rightharpoonup \mathbb{B}$ with:

$$\forall \varphi. (\vdash \varphi \leftrightarrow d_\mathcal{S}\, \varphi \downarrow \mathsf{tt}) \land (\vdash \neg\varphi \leftrightarrow d_\mathcal{S}\, \varphi \downarrow \mathsf{ff})$$

Moreover, because of consistency we have $d_\mathcal{S}\, \varphi \uparrow$ exactly if φ is an independent sentence.

- If $\mathcal{S}$ is complete, then $d_\mathcal{S}$ induces a decider for representable problems
- Even without completeness, $d_\mathcal{S}$ is a partial decider for representable problems...

Strongest Gödel à la Turing

Theorem

Every formal system representing K *has an independent sentence.*

Proof.

If $r : \mathbb{N} \to \mathbb{S}$ represents K, then $d_{\mathcal{S}} \circ r$ is a candidate decider for K. Thus there is some code c with $d_{\mathcal{S}}\,(r\ c) \uparrow$, so $r\ c$ is must be independent. $\square$

- Explicit independent sentence!
- No need to assume Markov's Principle!
- Still only applies to sound systems due to representability property...

Part 4
Strong Gödel à la Kleene

Recursive Inseparability

To avoid soundness, we would like that $c \in \overline{\mathsf{K}}$ implies $\vdash \neg r\, c$...

- $\overline{\mathsf{K}}$ is not semi-decidable, so can't be recognised in a formal system
- So we would like a semi-decidable subset of $\overline{\mathsf{K}}$
- Doesn't work for $\overline{\mathsf{K}}$ directly but there are other examples
- Recursive Inseparability: disjoint sets P, Q that are not separable by $d : X \to \mathbb{B}$

$$\forall x. (P\, x \to d\, x = \mathsf{tt}) \land (P\, x \to d\, x = \mathsf{ff})$$

Canonical Inseparable Sets

Lemma

The sets $\mathsf{K}^1 := \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow \mathsf{tt}\}$ *and* $\mathsf{K}^0 := \{c \in \mathbb{N} \mid \Theta_c\, c \downarrow \mathsf{ff}\}$ *are semi-decidable but recursively inseparable, in fact for every partial separation* $d : \mathbb{N} \rightharpoonup \mathbb{B}$ *with*

$$\mathsf{K}^1\, x \;\to\; d\, x \downarrow \mathsf{tt} \qquad \textit{and} \qquad \mathsf{K}^0\, x \;\to\; d\, x \downarrow \mathsf{ff}$$

one can construct a concrete value c *such that* $d\, c$ *diverges.*

Proof.

We first define a partial function $f : \mathbb{N} \rightharpoonup \mathbb{B}$ diagonalising against s by:

$$f\, x := \begin{cases} \xi_{\mathsf{tt}} & \text{if } d\, x \downarrow \mathsf{ff} \\ \xi_{\mathsf{ff}} & \text{if } d\, x \downarrow \mathsf{tt} \\ \xi_{\bot} & \text{otherwise} \end{cases}$$

Now using EPF we obtain a code c for f and deduce that $d\, c \uparrow$ by similar equivalences. $\square$

Strong Gödel à la Kleene

We say that a formal system $\mathcal{S}$ separates sets $P, Q \subseteq X$ if there is a function $r : X \to \mathbb{S}$ with

$$\forall x.\, (x \in P \to \vdash r\, x) \wedge (x \in Q \to \vdash \neg r\, x).$$

Theorem

Every formal system separating K^1 and K^0 has an independent sentence.

Proof.

If $r : \mathbb{N} \to \mathbb{S}$ separates K^1 and K^0, then $d_{\mathcal{S}} \circ r$ is a partial separation of K^1 and K^0. Thus there is some code c with $d_{\mathcal{S}}\,(r\, c)\uparrow$, so $r\, c$ is must be independent. □

Corollary

If $\mathcal{S}$ separates K^1 and K^0, then every extension $\mathcal{S}' \supseteq \mathcal{S}$ has an independent sentence.

Part 5

Instantiation to FOL

Essential Incompleteness of Robinson's Q

To instantiate these abstract proofs to Q, we need a stronger assumption than EPF:

Axiom ($\mathsf{CT_Q}$)

For every $f : \mathbb{N} \rightharpoonup \mathbb{B}$ there is a Σ_1-formula φ with: $f\,x \downarrow b \;\leftrightarrow\; \mathsf{Q} \vdash \forall b'.\, \varphi(\overline{x}, b') \leftrightarrow b' = \overline{b}$

$\mathsf{CT_Q}$ implies that Q and every consistent extension of it has an independent sentence:

- $\mathsf{CT_Q}$ implies EPF
- $\mathsf{CT_Q}$ implies that Q separates the problems K^1 and K^0
- Claim follows from the abstract incompleteness result

Deriving $\mathsf{CT_Q}$ from Church's Thesis

$\mathsf{CT_Q}$ is implied by a more conventional formulation of Church's thesis:

Axiom (CT)

Every $f : \mathbb{N} \rightharpoonup \mathbb{B}$ is computable by a μ-recursive function.

Instead of showing directly that Q represents μ-recursive functions, we chained a few results that already had been mechanised in Coq:

1. Equivalence of several models of computability (Forster, 2021b)
2. DPRM theorem (Larchey-Wendling and Forster, 2019): computability is Diophantine
3. Diophantine constraints can be embedded into Q (Kirst and Hermes, 2021)
4. Use Rosser's trick to obtain representability as in $\mathsf{CT_Q}$

Conclusion

Summary: 5 Shades of Gödel

1. Post's Theorem: complete systems are decidable
2. Posts's Theorem: systems representing undecidable problems are incomplete
3. Post's Theorem + Halting Problem: systems representing K are incomplete
4. Refined Post's Theorem + Halting Problem: systems representing K have gaps
5. Refined Post's Theorem + Recursive Insep.: systems separating K^1 and K^0 have gaps

Easy computational arguments, elegant formalisation in constructive logic!

Literature Pointers

- Richman (1983): Church's Thesis without Tears
- Smith (2021): Gödel without (too many) Tears
- Kirst and Peters (2023): Gödel's Theorem without Tears

- Bauer (2017): Five Stages of Accepting Constructive Mathematics
- Bauer (2006): First Steps in Synthetic Computability Theory
- Forster (2022): Parametric Church's Thesis: Synthetic Computability without Choice

- Kleene (1951): A Symmetric form of Gödel's Theorem
- Kleene (2002): Mathematical Logic
- Beklemishev (2010): Gödel Incompleteness Theorems and the Limits of their Applicability

Take-Home Messages

1. Even incompleteness of Rosser's strength is nothing but basic computability theory!
2. Constructive logic is a great framework to explore computability theory!
3. With the right setup, deep theorems can be mechanised in 200 lines!

Thank You!!!

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