

# Propositions and Types, Proofs and Programs

## Part II: Natural Deduction

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ANU Logic Summer School 2024

# A Quick Recap of Natural Deduction

Natural deduction derivations form a **tree of formulae**

- We will see that they can also be presented ‘sequent-style’, which means to keep track of our assumptions at all times

The formulae on the leaves are ‘alive’ or ‘dead’

- If alive, they are **assumptions**
- If dead we write them inside square brackets. They are **discharged** assumptions from a subproof, ‘killed’ by one of our proof rules.

# The Structure of Natural Deduction

The simplest natural deduction proof is

A

This means 'from the assumption A, conclude A'

- No proof rules were involved

A larger derivation might look like something like:

$$\frac{\frac{\frac{C}{\quad} \quad \frac{[A] \quad B}{\quad}}{\quad} \quad D}{\quad} \quad E}{\quad} \quad F$$

"From assumptions B and C (not A!), conclude F"

# The Rules of Natural Deduction

Rules may have any number of premises, but only 1 conclusion

Rules **introduce** or **eliminate** a particular connective

- The introduction rules should follow the BHK interpretation, and resemble sequent calculus **right** rules
- The elimination rules should be the introduction rules, ‘upside down’ (we will not make this condition formal)

Some rules can ‘kill’ formulae in the leaves

- If a formula  $A$  is written in square brackets above a premise in a rule, it means ‘you may kill any number (0, 1, or more) of occurrences of  $A$  in the leaves’.
- You do not have to kill all occurrences of such an  $A$

# Conjunction

*“A proof of  $A \wedge B$  is given by presenting a proof of  $A$  and a proof of  $B$ ”*

$$\frac{A \quad B}{A \wedge B} \wedge I$$

If we have a proof of  $A \wedge B$  then we can eliminate it to get a proof of either  $A$ , or if we prefer,  $B$ :

$$\frac{A \wedge B}{A} \wedge E_1 \quad \frac{A \wedge B}{B} \wedge E_2$$

# Conjunction Example

From  $A \wedge B$ , conclude  $B \wedge A$ :

$$\begin{array}{c} \wedge E_2 \quad \frac{A \wedge B}{B} \quad \frac{A \wedge B}{A} \quad \wedge E_1 \\ \frac{B \quad A}{B \wedge A} \quad \wedge I \end{array}$$

# False

“ $\perp$  has no proof”

- So  $\perp$  has no introduction rule

But it *does* have an elimination rule: if we have somehow concluded  $\perp$ , then anything goes!

$$\frac{\perp}{A} \quad \perp E$$

Simple exercise: from  $A \wedge \perp$ , prove B

# Implication

*“A proof of  $A \rightarrow B$  is a construction which permits us to transform any proof of  $A$  into a proof of  $B$ ”*

$$\frac{[A] \quad B}{A \rightarrow B} \rightarrow I$$

If we have a proof of  $A \rightarrow B$  and a proof of  $A$ , then we combine them:

$$\frac{A \rightarrow B \quad A}{B} \rightarrow E$$



# Implication Example

$A \rightarrow (B \rightarrow A)$  without assumptions:

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \rightarrow A \end{array}}{A \rightarrow (B \rightarrow A)} \rightarrow I$$

# Implication Example

$A \rightarrow (B \rightarrow A)$  without assumptions:

$$\begin{array}{c} [A, B] \\ \vdots \\ A \\ \hline B \rightarrow A \quad \rightarrow\text{I} \\ \hline A \rightarrow (B \rightarrow A) \quad \rightarrow\text{I} \end{array}$$

# Implication Example

$A \rightarrow (B \rightarrow A)$  without assumptions:

$$\frac{\frac{[A]}{B \rightarrow A} \rightarrow I}{A \rightarrow (B \rightarrow A)} \rightarrow I$$

B is never used, so does not need to be killed.

# Implication Example II

$(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$  without assumptions:

$$\begin{array}{l} \rightarrow E \quad \frac{[A \rightarrow (B \rightarrow C)] \quad [A] \quad [A \rightarrow B] \quad [A]}{\quad} \rightarrow E \\ \quad \quad \quad \frac{B \rightarrow C \quad B}{C} \rightarrow E \\ \quad \quad \quad \frac{C}{A \rightarrow C} \rightarrow I \\ \quad \quad \frac{A \rightarrow C}{(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)} \rightarrow I \\ \quad \frac{(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)}{(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))} \rightarrow I \end{array}$$

# Disjunction

*“A proof of  $A \vee B$  is given by presenting either a proof of  $A$  or a proof of  $B$ ”*

$$\frac{A}{A \vee B} \text{VI}_1 \qquad \frac{B}{A \vee B} \text{VI}_2$$

For elimination, we cannot always know which introduction rule was used to get  $A \vee B$ , so we consider both possibilities:

$$\frac{A \vee B \quad \begin{array}{cc} [A] & [B] \\ C & C \end{array}}{C} \text{VE}$$

# Disjunction Example

From  $A \vee B$ , conclude  $B \vee A$ :

$$\frac{A \vee B \quad \text{VI}_2 \quad \frac{[A]}{B \vee A} \quad \frac{[B]}{B \vee A} \quad \text{VI}_1}{B \vee A} \quad \text{VE}$$

# All Natural Deduction Rules

$$\frac{A \quad B}{A \wedge B}$$

$$\frac{A \wedge B}{A}$$

$$\frac{A \wedge B}{B}$$

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

$$\frac{A \vee B \quad [A] \quad C \quad [B] \quad C}{C}$$

$$\frac{[A] \quad B}{A \rightarrow B}$$

$$\frac{A \rightarrow B \quad A}{B}$$

$$\frac{\perp}{A}$$

# Natural Deduction, Sequent Style

It is sometimes convenient to record all our current assumptions in every node of the tree

- Does clutter our proofs, but can help us keep track

We do this via a (single conclusion) sequent  $\Gamma \vdash A$

- This is *not* sequent calculus; just a different notation for natural deduction
- $\Gamma$  is a multiset of formulae, written as a list but whose order is irrelevant

Our assumptions can then be read off the left hand side of the root, rather than from the leaves.

Killing an assumption means removing it from our left hand side.



# Sequent Style: Assumption Rule

Because we are building a tree of sequents, not formulae, we need to change our leaves from lone formulae to explicit uses of assumptions:

$$\frac{}{\Gamma, A \vdash A} \text{AX}$$

This is a 0-premise rule, i.e. an axiom.

Because we do not care about the ordering of the left hand side,  $A$  could appear anywhere in our representation of  $\Gamma$ .

# Sequent Style: Killing Assumptions

Our two rules that kill assumptions:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \quad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Because  $\Gamma$  is a multiset (can contain more than one occurrence of a formula) killing  $A$  does not mean  $A$  cannot still be in  $\Gamma$ .

- This is necessary to prove e.g.  $A \rightarrow (A \rightarrow A)$

# All Sequent-Style Natural Deduction Rules

$$\frac{}{\Gamma, A \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\Gamma \vdash A \vee B$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$\Gamma \vdash C$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

$$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash A}$$

# Sequent-Style Natural Deduction Example

$$\begin{array}{c}
 \frac{A \rightarrow B, \mathbf{A \rightarrow (B \rightarrow C)}, A \vdash \mathbf{A \rightarrow (B \rightarrow C)} \quad A \rightarrow B, A \rightarrow (B \rightarrow C), \mathbf{A \vdash A}}{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash B \rightarrow C} \quad \frac{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash \mathbf{A \rightarrow B} \quad A \rightarrow B, A \rightarrow (B \rightarrow C), \mathbf{A \vdash A}}{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash B} \\
 \hline
 \frac{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash B \rightarrow C \quad A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash B}{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash C} \\
 \hline
 \frac{A \rightarrow B, A \rightarrow (B \rightarrow C), A \vdash C}{A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash A \rightarrow C} \\
 \hline
 \frac{A \rightarrow B, A \rightarrow (B \rightarrow C) \vdash A \rightarrow C}{A \rightarrow B \vdash (A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)} \\
 \hline
 \vdash (A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))
 \end{array}$$

The extra book-keeping gets a bit heavy at the top!

- But it does help us to understand the ‘state of play’ at any point of the tree

# An Aside on Classical Natural Deduction

Sequent calculus seems equally good for classical and intuitionistic logic

- Multiple conclusion vs single conclusion

But, with natural deduction, classical logic does not feel so good.

Extension requires a double negation elimination rule, or similar

- this 'breaks' the principle of using only introduction and elimination rules

We can also work with multiple conclusions on the right

- Not obvious how to do this correctly! Took until the 1990s to work out
- And what might this mean for programming purposes?

# Organising our Proofs

One theorem might have many equally good proofs.

- e.g. to prove  $(A \wedge A) \rightarrow A$ , do we choose to go via  $\wedge E_1$  or  $\wedge E_2$ ?

But that does not mean *every* proof is equally good!

What if our proof contained this ‘roundabout’ argument?

$$\frac{\frac{A \quad B}{A \wedge B} \wedge I}{A} \wedge E_1$$

We’ve done work to conclude  $A$  from  $A$ , and should cut this part out.

- This process can be done algorithmically, and is called **normalisation**

# Towards Normalisation

The concept of normalisation was invented by Gentzen and perfected by **Dag Prawitz** (1936-) without concern for computer science.

But it is exactly this concept that animates the Curry-Howard isomorphism that our lectures are driving towards.

We will therefore briefly put logic to one side to recap the main idea on the programming side of our picture: **lambda-calculus**.

Photo c/o [Stockholm University](#)

