Qualitative Calculi via Relation Algebras. Part I

Tomasz Kowalski

ANU Logic Summer School

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About these lectures

- Lecturer: Tomasz Kowalski
  - Jagiellonian University, Kraków (full time job)
  - La Trobe University, Melbourne (adjunct prof)
  - University of Queensland, Brisbane (honorary)
- email: `tomasz.s.kowalski@uj.edu.pl`
- web: `https://filozofia.uj.edu.pl/tomasz-kowalski/`
  `https://iphils.uj.edu.pl/~t.kowalski/`
- Interrupt at any time with questions.
- There will be very few proofs. I want to focus on ideas.
- There will be some exercises. I am happy to discuss them during the breaks and any other time you can buttonhole me.
- If you like, you may email your solutions to me.

Some background literature

- R. Hirsch and I. Hodkinson, *Relation algebras by games*, Elsevier, 2002.
- R. Maddux, *Relation algebras*, Elsevier, 2006.
- F. Dylla *et al.*, A Survey of Qualitative Spatial and Temporal Calculi: Algebraic and Computational Properties, *ACM Computing Surveys*, vol. 50, no. 1 (2017), 1–39.
- R. Hirsch, M. Jackson and TK, Algebraic foundations for qualitative calculi and networks, *Theoret. Comput. Sci.* **768** (2019), 99–116.
- A. Inants and J. Euzenat, So, what exactly is a qualitative calculus?, *Artificial Intelligence* **289** (2020), 1–14.
- B. Al-Juaid *et al.*, Edge colourings and qualitative representations of chromatic algebras, *J. Algebraic Combin.* **58** (2023), no. 1, 157–182.

Qualitative calculi

Intuitions and some examples

A riddle to begin with

Brothers and sisters have I none, but this man's father is my father's son.

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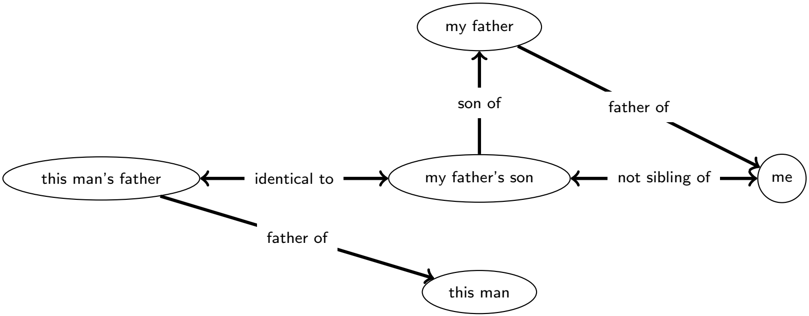
Brothers and sisters have I none, but this man's father is my father's son. Take a binary constraint network.



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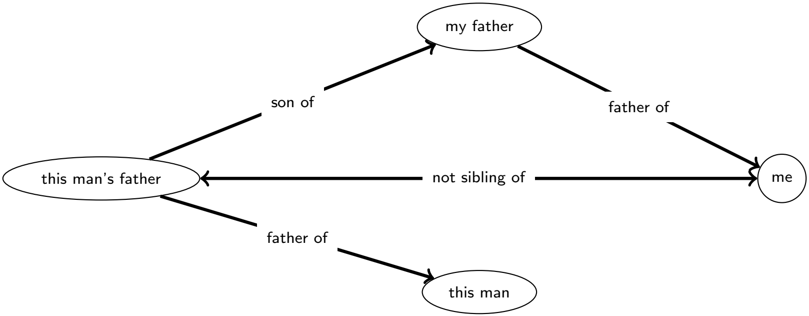

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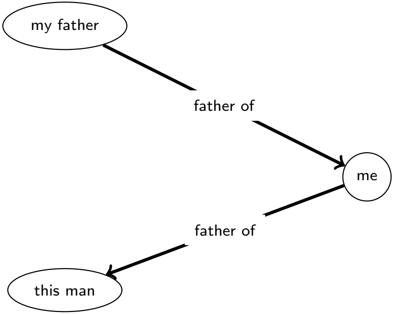
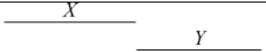
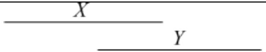
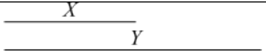
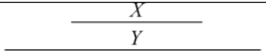
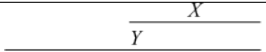
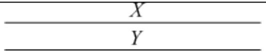


and refine. And refine again.

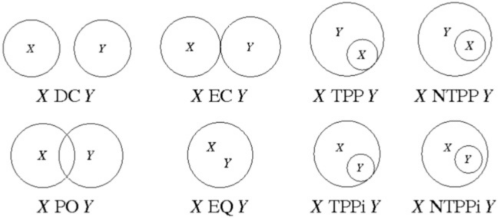
Serious example 1: Allen's interval algebra

13 atomic relations between time intervals

	X before Y	Y after X
	X meets Y	Y is met by X
	X overlaps with Y	Y is overlapped by X
	X starts Y	Y is started by X
	X during Y	Y contains X
	X finishes Y	Y is finished by X
	X equals Y	Y equals X

Serious example 2: RCC8

8 atomic relations between regions of a plane



X DC Y	X is disconnected from Y	Y DC X
X EC Y	X is externally connected to Y	Y EC X
X TPP Y	X is a tangential proper part of X	Y TPPi X
X NTPP Y	X is a non-tangential proper part of Y	Y NTPPi X
X PO Y	X properly overlaps Y	Y PO X
X EQ Y	X is equal to Y	Y EQ X

Serious example 3: atom tables and the point algebra

- Let C be an infinite linear order without endpoints.
- Consider relations $=$, $<$ and $>$ in $C \times C$. Their composition table is given below, with 1 standing for the total relation.

;	$=$	$<$	$>$
$=$	$=$	$<$	$>$
$<$	$<$	$<$	1
$>$	$>$	1	$>$

- Note that all information about the subalgebra of $\mathcal{P}(C \times C)$ generated by $=$, $<$ and $>$ can be read off this table.
- In particular, $=$, $<$, $>$ are atoms, partitioning $C \times C$.
- But the "partition trick" will not work in general. Why?

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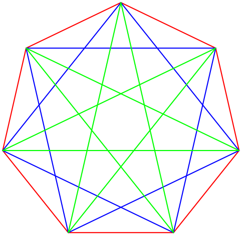
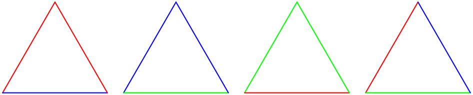
Serious example 4: colourings



$\circ$	$=$	red	green	blue
$=$	$=$	r	g	b
r	r	$=, b$	g, b	r, g
g	g	g, b	$=, r$	r, b
b	b	r, g	r, b	$=, g$

We say that the triangles below are consistent.



All other triangles are forbidden.

Enter relation algebras

Notation

We deal with abstract algebras and concrete representations as binary relations and separate the notation, to some extent.

- Abstract: we use $\wedge$, $\vee$ and $\neg$ as the basic boolean operators. The identity constant is 1', the converse operator is $\breve{}$ and any algebraic multiplication-like operator is ;.
- Concrete: we write $\cap$, $\cup$ and $^{-}$, for the operators corresponding to $\wedge$, $\vee$ and $\neg$. We write Id for the identity relation, corresponding to the abstract 1', The converse (inverse) of r is written as r^{-1}. Composition is written as $r \circ s$.

That is, in short:

abstract	$\vee$	$\wedge$	$\neg$	1'	$\breve{}$	;
concrete	$\cup$	$\cap$	$^{-}$	Id	$^{-1}$	$\circ$

Algebras of binary relations

Let U be any set. Consider $\mathcal{P}(U \times U)$ with the following operations:

- union ($\cup$), intersection ($\cap$) and complement ($^{-}$)
- relational composition ($\circ$) and converse ($^{-1}$)
- identity relation (Id), bottom ($\emptyset$), top ($U \times U$)

The structure $\langle \mathcal{P}(U \times U); \cup, \cap, \circ, ^{-}, ^{-1}, Id, \emptyset, U \times U \rangle$ is an algebra of binary relations.

- A. De Morgan, *On the syllogism IV: and on the logic of relations* (1858)
- Ch.S. Peirce, *Note B: the logic of relatives* (1883)
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Algebras of binary relations as a class (overview)

- The class of all algebras of binary relations is not closed under direct products or subalgebras. It is closed under homomorphic images (because every such algebra is simple).
- But if we modify the definition slightly, by changing the universe from $\mathcal{P}(U \times U)$ to $\mathcal{P}(E)$, where E is any equivalence relation on U, we get a class that is closed under direct products.
- If we take subalgebras of algebras of relations on the universe $\mathcal{P}(E)$, we get a class closed under H, S, P, that is, a variety.
- It is the variety RRA of representable relation algebras.
- But RRA is not finitely axiomatisable.
- A natural finitely axiomatisable variety properly containing RRA is the variety RA of relation algebras.

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More detail: equations satisfied by alg's of binary rel's

1. Equations making $\langle \mathcal{P}(U \times U); \cup, \cap, ^{-}, Id, \emptyset, U \times U \rangle$ into a Boolean algebra.
2. Equations making $\langle \mathcal{P}(U \times U); \circ, ^{-1}, Id \rangle$ into an involutive monoid.
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   - $x \circ (y \cup z) = x \circ y \cup x \circ z$, $(x \cup y) \circ z = x \circ z \cup y \circ z$
   - $x \circ \emptyset = \emptyset = \emptyset \circ x$
   - $(x \cup y)^{-1} = x^{-1} \cup y^{-1}$
   - $\emptyset^{-1} = \emptyset$
4. Triangle (Peircean) laws:

$$x \circ y \cap z = \emptyset \text{ iff } x^{-1} \circ z \cap y = \emptyset \text{ iff } z \circ y^{-1} \cap x = \emptyset.$$

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More detail: do these axiomatise our class?

Definition (Jónsson and Tarski, 1948)

- An (abstract) relation algebra (RA) is any algebra $\mathbf{A} = \langle A; \vee, \wedge, ;, \neg, \breve{}, 1', 0, 1 \rangle$ satisfying equations from the previous slide.
- An (abstract) RA isomorphic to a subalgebra of a direct product of (concrete) algebras of binary relations is called a representable relation algebra (RRA).

Question (Jónsson and Tarski, 1948)

Is every relation algebra representable? Or, is every RA a RRA?

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More detail: non-representable RAs

Theorem (Lyndon, 1950)

There are non-representable relation algebras.

- Lyndon's counterexample is somewhat big (2^{52} elements).
- McKenzie (1974), found a minimal one. Here it is:

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More detail: non-finitely based variety

Theorem (Tarski, 1955)

The class RRA *is a variety.*

Theorem (Monk, 1964)

The variety RRA *is not finitely axiomatised in FOL.*

Proof strategy.

- Construct infinitely many non-representable algebras, say $\mathbf{M}_n$ ($n \in \mathbb{N}$), such that
- their ultraproduct $\prod_{n \in \mathbb{N}} \mathbf{M}_n / U$ for a suitable U turns out to be representable.
- Hence, the complement of RRA is not closed under ultraproducts. □

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Theorem (Tarski, 1955)

The class RRA *is a variety.*

Theorem (Monk, 1964)

The variety RRA *is not finitely axiomatised in FOL.*

Proof strategy.

- Construct infinitely many non-representable algebras, say $\mathbf{M}_n$ ($n \in \mathbb{N}$), such that
- their ultraproduct $\prod_{n\in\mathbb{N}} \mathbf{M}_n / U$ for a suitable U turns out to be representable.
- Hence, the complement of RRA is not closed under ultraproducts. □

More detail: finite rep's and representation undecidability

- There are finite RRAs without finite representation, for example, the point algebra.

Theorem (Hirsch and Hodkinson, 1999)

It is undecidable whether a finite RA **A** *is representable.*

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- Use equivalence of representability to a winning strategy in a certain game.
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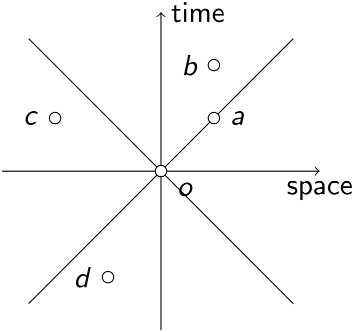
A relativistic version of the point algebra



Take the following relations between spacetime points:

- $1'(x, x)$: identity,
- $s(o, c)$: spacelike separation,
- $\ell(o, a)$: lightlike separation,
- $f(b, o)$: b in future cone of o,
- $p(d, o)$: d in past cone of o.

Exercise 1.

Write a composition table for the relation algebra with atoms $1'$, s, ℓ, f and p.

Exercises: associativity of composition

The following sentence says that composition of binary relations is associative.

$$\forall x,y \exists z_1,z_2,z_3,z_4: \quad R(x,z_1) \wedge S(z_1,z_2) \wedge T(z_2,y) \iff R(x,z_3) \wedge S(z_3,z_4) \wedge T(z_4,y).$$

It is a theorem of first-order logic.

Exercise 2.

Write an equivalent sentence using only 3 variables. It does not need to be in a prenex form. In fact, it cannot be.

Exercise 3**.

Can you prove the sentence above in first-order logic using only 3 variables?

Qualitative representations

The annulus problem in RCC8

- Naive calculation with disks suggests: $EC \circ EC = DC \cup EC \cup PO \cup TPP \cup TPPi \cup EQ$. In particular, $EC \subset EC \circ EC$.
- But if one region is completely surrounded by another, as in the right-hand side picture below, where region y is the annulus surrounding region x, then we have $xECy$, but $(x, y) \notin EC \circ EC$.

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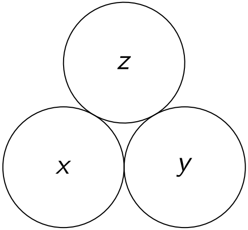

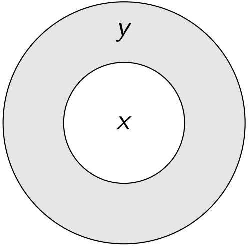

Weak composition: first attempt

- As a work around, weak composition has been used:

$$R\,;S = \bigvee\{A \in At(T) \colon (R \circ S) \cap A \neq \emptyset\}$$

  where $At(T)$ stands for an atomic relation, that is, a partition class.
- But this depends on having an atomic and complete Boolean subalgebra B of $\mathfrak{Re}(U)$ for some set U.
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- *Aside: if you are familiar with modal logic, it may remind you of finest filtration.*

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Herds of binary relations and weak composition

Definition

Let D be a set and let $\mathcal{S}$ be a set of binary relations over D, that is, $\mathcal{S} \subseteq \mathcal{P}(D \times D)$. $\mathcal{S}$ is a herd if

1. $\mathcal{S}$ forms a boolean set algebra with top element $D \times D$.
2. $\mathrm{Id} \in \mathcal{S}$,
3. If $A \in \mathcal{S}$ then the converse relation A^{-1} is in $\mathcal{S}$.

- In a herd $\mathcal{S}$ given any two elements $A, B \in \mathcal{S}$ if there is a minimal $C \in \mathcal{S}$ containing $A \circ B$ then we say that the weak composition of A and B is C.
- If $\mathcal{S}$ is finite, then such a minimal element is sure to exist, since $\mathcal{S}$ is closed under finite intersections.

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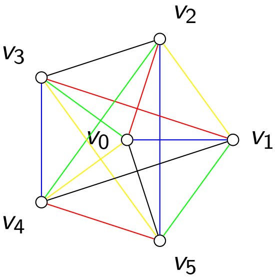
Weak composition is not associative

- Consider the set $\{v_0, v_1, v_2, v_3, v_4, v_5\}$ and the relations 0 (identity), 1 (blue), 2 (red), 3 (green), 4 (yellow), 5 (black):



$$i \,;\, j = \begin{cases} 0 & \text{if } i = j \\ (i \cup j)^{-} & \text{if } i \neq j \end{cases}$$

- These relations form a herd. Why?
- Weak composition is defined by the formula on the right.

INDU: Allen's INtervals plus DUration

Atomic relation	$<$	$>$	$=$	Atomic relation	$<$	$>$	$=$
X before Y				X after Y			
X meets Y				X is met by Y			
X overlaps with Y				X is o'lapped by Y			
X starts Y				X is started by Y			
X finishes Y				X is finished by Y			
X during Y				X contains Y			
$X = Y$	✗	✗	✓				

Exercise 4.

Which combinations are possible in the table above? Try without googling.

Exercise 5*.

INDU algebra is not associative. Find all associative triples of atomic relations.

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Nonassociative relation algebras

Definition (Maddux, 1982)

A nonassociative relation algebra (NA) is an algebra $\mathbf{A} = \langle A; \vee, \wedge, ;, \neg, \breve{}, 1', 0, 1 \rangle$ such that

1. $\langle A; \vee, \wedge, \neg, 1', 0, 1 \rangle$ is a Boolean algebra.
2. $\langle A; ;, \breve{}, 1' \rangle$ is an involutive groupoid with unit.
3. The following further equations hold:
   - $x ; (y \vee z) = x ; y \vee x ; z$, $(x \vee y) ; z = x ; z \vee y ; z$
   - $x ; 0 = 0 = 0 ; x$
   - $(x \vee y)\breve{} = x\breve{} \vee y\breve{}$
   - $0\breve{} = 0$
4. Triangle (Peircean) laws:

$$x ; y \wedge z = 0 \text{ iff } x\breve{} ; z \wedge y = 0 \text{ iff } z ; y\breve{} \wedge x = 0.$$

Qualitative representation

Definition

Let **A** be a non-associative algebra. A qualitative representation ϕ of an algebra **A** is an injection to a herd $\mathcal{S}$ of binary relations over base D, such that

1. $0^\phi = \varnothing$, $1^\phi = D \times D$, $(1')^\phi = \mathsf{Id}_D$,
2. $(a \vee b)^\phi = a^\phi \cup b^\phi$, $(\neg a)^\phi = (D \times D) \setminus a^\phi$,
3. $(a^\smile)^\phi = (a^\phi)^{-1}$,
4. $c^\phi \supseteq a^\phi \circ b^\phi \iff c \geq a \,;\, b$

for all $a, b, c \in A$. If **A** has a qualitative representation, then we say that **A** is a qualitatively representable algebra. The class of all qualitatively representable algebras we will denote by QRA.

- If $(a \,;\, b)^\phi = a^\phi \circ b^\phi$ for all $a, b \in \mathcal{A}$ then the qualitative representation ϕ is a strong representation.

Qualitative representations vs. representations

What they really are

Qualitative rep's vs rep's. Two slogans

Note that

- $(a\,;b)^\phi = a^\phi \circ b^\phi$ implies $c^\phi \supseteq a^\phi \circ b^\phi \iff c \geq a\,;b$, so every representation is a qualitative representation.
- The converse is not true.

Representation slogan

In a representation every consistent triangle occurs everywhere it can occur.

Qualitative representation slogan

In a qualitative representation, every consistent triangle occurs somewhere.

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Ramsey-like examples

- For each n consider a RA $\mathbf{M}_n$ on $n+1$ atoms $1', a_1, \ldots, a_n$, whose composition table is given by:

$$1' \,;\, a_i = a_i = a_i \,;\, 1' \quad \text{and} \quad a_i \,;\, a_j = \begin{cases} 0' & \text{if } i \neq j \\ a_i^- & \text{if } i = j \end{cases}$$

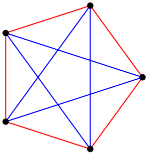
  so that monochromatic triangles are forbidden, but all non-monochromatic triangles are allowed.

- In $\mathbf{M}_n$ (with $a_1 = b$ and $a_2 = r$) we have

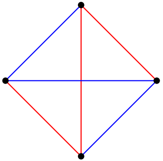
;	$1'$	b	r
$1'$	$1'$	b	r
b	b	$1' \vee r$	$0'$
r	r	$0'$	$1' \vee b$

Ramsey examples: $\mathbf{M}_2$

- By the simplest case of Ramsey Theorem, any (strong) representation of $\mathbf{M}_2$ must not have more than 5 points.

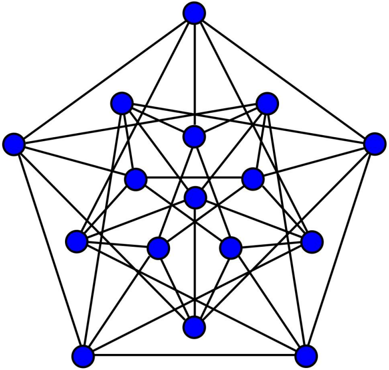


- But, it cannot have fewer than 5 points, because every blue edge must be the base of 3 triangles.
- For a qualitative representation 4 points suffice.



Ramsey examples: $\mathbf{M}_3$

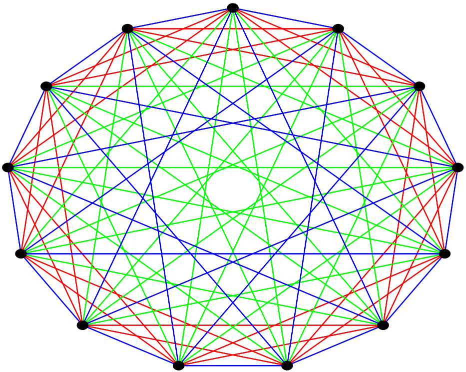
- Here the Ramsey number is 16. So you may expect a representation on K_{16}. Indeed, you get one.
- To prove it, you can play around with the Clebsch graph



- Or you can play around with $GF(16)$

Ramsey examples: rep's of $\mathbf{M}_3$

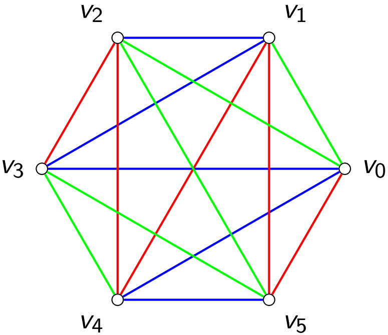
- Can a representation be any smaller?
- Yes, on K_{13}. It can be shown that nothing smaller will do.



- Interestingly, neither will K_{14} and K_{15}.

Ramsey examples: qualitative rep's of $\mathbf{M}_3$

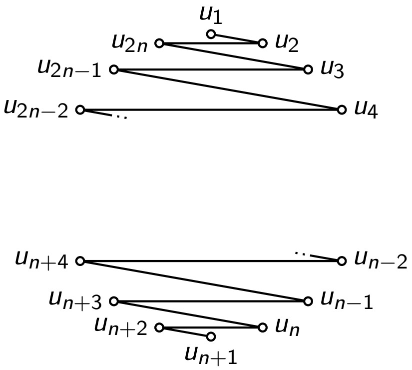
- A qualitative representation of $\mathbf{M}_3$:



- You can get a qualitative representation on K_5, but the one above has a method to it.

Ramsey examples: qualitative rep's of $\mathbf{M}_n$

- We get a representation of $\mathbf{M}_n$ as follows.
- Start with one colour:



- Rotate, change colour, and repeat.

Ramsey examples: a theorem and a half

Theorem (Al-Juaid, Jackson, Koussas, TK)

The algebras $\mathbf{M}_n$ are qualitatively representable for any $n \geq 2$.

Representability

Representations exist up to $\sim$ 4000 colours (J. Alm), with possible exceptions of 8 and 13. No general theorem.

Exercise 6*.

Find a representation of $\mathbf{M}_8$, or prove that none exists.

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