Qualitative Calculi via Relation Algebras. Part II

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In this part we assume that all algebras are simple. This is just for convenience... mainly.

Networks and games

Pre-networks and networks

Definition

Let **A** be a nonassociative relation algebra.

- A pre-network over **A** is a nonempty, *complete* directed graph N with each edge (x, y) labelled by an element $N(x, y)$ of **A**.
- A network over **A** is a pre-network, satisfying the consistency conditions
  1. $N(x, x) \leq 1'$,
  2. $(N(x, y) \,;\, N(y, z)) \wedge N(x, z) \neq 0$,

  for all nodes $x, y, z \in N$.
- An atomic network is a network such that each edge is labelled by an atom of **A**.
- A network N is strict, if $N(x, y) \leq 1'$ implies $x = y$.

Refinement

Definition

Let N_i and N_j be pre-networks over an algebra $\mathbf{A}$. We say that N_j refines N_i, and write $N_i \leq N_j$, if

- the vertices of N_i are a subset of the vertices of N_j, and
- for any $(x, y) \in N_i \times N_i$ we have $\mathbf{A} \models N_j(x, y) \leq N_i(x, y)$.

Exercise 8

Let N_i be a network, and let N_j be a pre-network that refines N_i. Is N_j a network?

Hirsch & Hodkinson Games

Definition (Game setup)

Let $n \leq \omega$. For an algebra $\mathbf{A}$ and a pre-network over $\mathbf{A}$, we define a game $\mathcal{G}_n(N, \mathbf{A})$ in which two players, $\forall$ and $\exists$, construct a sequence of pre-networks over $\mathbf{A}$:

$$N = N_0 \leq N_1 \leq N_2 \leq \cdots \leq \begin{cases} N_n & \text{if } n < \omega, \\ N_n \leq \ldots & \text{if } n = \omega. \end{cases}$$

Definition (Moves of $\mathcal{G}_n(N, \mathbf{A})$)

Assume t is the current round, that is, we have a pre-network N_t. Then $\forall$ and $\exists$ move as follows.

- $\forall$ chooses elements $x, y \in N_t$, and elements $r, s \in A$.
- $\exists$ responds with a pre-network $N_{t+1} \geq N_t$, such that either of the following holds:
  - (R) N_{t+1} is the same as N_t except that $N_{t+1}(x, y) = N_t(x, y) \wedge (r \,;\, s)^-$.
  - (A) N_{t+1} has one node z in addition to those of N_t and the labels on the edges of N_{t+1} are as follows:
    - $N_{t+1}(x, z) = r$, $N_{t+1}(z, z) = 1'$, $N_{t+1}(z, y) = s$
    - $N_{t+1}(x, y) = N_t(x, y) \wedge (r \,;\, s)$
    - for any $(x', y') \in N_t \times N_t$, if $(x', y') \neq (x, y)$, then $N_{t+1}(x', y') = N_t(x', y')$.
    - all other edges of N_{t+1} are labelled by 1.

Winning

Definition

- $\exists$ wins $\mathcal{G}_n(N, \mathbf{A})$, if every pre-network constructed during the game is a network (i.e., satifies consistency conditions). Otherwise, $\forall$ wins.
- $\exists$ has a winning strategy for $\mathcal{G}_n(N, \mathbf{A})$ if she can win regardless of the moves $\forall$ makes.

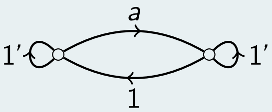
- Intuitively, $\exists$ having a winning strategy means she can survive $\mathcal{G}_n(N, \mathbf{A})$ without running into a contradiction. If $n = \omega$, she can survive forever.
- Even if $\exists$ wins, the final network constructed may not be strict. But this is not a problem, as we can quotient out what we need.

Representability

Representation of RAs by games

Theorem (Hirsch, Hodkinson)

Let $\mathbf{A}$ *be an at most countable, simple relation algebra. For any* $a \in A$ *we let* I_a *be a network*



Then, the following are equivalent

1. $\exists$ *has a winning strategy in any game* $\mathcal{G}_\omega(I_a, \mathbf{A})$.
2. $\mathbf{A}$ *has a square representation.*
3. $\mathbf{A}$ *is representable.*

Sketch of proof

Lemma (H & H)

Let N be a network over $\mathbf{A}$. *The relation* $\sim$ *on N defined by* $x \sim y$ *iff* $N(x, y) \leq 1'$ *is a congruence. The quotient* $N/_\sim$ *is a strict network.*

Proof.

- Since $\exists$ has a winning strategy, she can survive forever.
- The final network may not be strict, but by the lemma above, we make it strict, and get a representation. □

Comments

- By universal algebra, a RA **B** is representable iff every countable subalgebra of **B** is, so the restriction to countable **A** is harmless.
- Since every RA is isomorphic to a direct product of simple RAs, the restriction to simple RA **A** is harmless.
- Even if **A** is finite and $\exists$ has a winning strategy, we may still need an infinitely long game.

Qualitative representability games

- There are many ways of modifying Hirsch & Hodkinson games to match the weaker notion of qualitative representability.
- I will present the one I like, but it is neither better nor worse than others.

Intuition.

- During the game we construct consistent triangles.
- For RAs, $\exists$ must be able to build a consistent triangle on any edge with which it is consistent.
- For QRAs, $\exists$ may dodge a little and only build a consistent triangle on another edge of her choice.

Definition (Moves of $\mathcal{QG}_n(N, \mathbf{A})$)

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- $\forall$ chooses elements $x, y \in N_t$, and elements $r, s \in A$, such that $\mathbf{A} \models N(x, y) \wedge r \,;\, s \neq 0$.
- $\exists$ responds with a pre-network $N_{t+1} \geq N_t$ and elements $u, v \in N_t$, such that:

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  - $N_{t+1}(u, v) \leq N_{t+1}(x, y)$
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Winning is defined exactly as before

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Qualitative representability by games

Theorem

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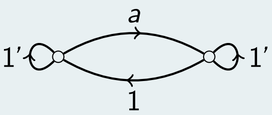

Then, the following are equivalent

1. $\exists$ *has a winning strategy in any game* $\mathcal{QG}_\omega(I_a, \mathbf{A})$.
2. **A** *has a qualitative representation.*

- Unlike RAs, finite QRAs are finitely representable.

Qualitative representability for finite algebras

Lemma (Hirsch, Jackson, TK)

Let $\mathbf{A}$ *be a finite NA. The following are equivalent.*

- $\mathbf{A}$ *has a qualitative representation.*
- *There is an atomic network* N *over* $\mathbf{A}$ *such that for each consistent triple of atoms* (a, b, c) *of* $\mathbf{A}$ *there are nodes* $x, y, z \in N$ *such that* $N(x, y) = a$, $N(y, z) = b$ *and* $N(x, z) = c$.

Lemma (Hirsch, Jackson, TK)

If $\mathbf{A}$ *is a qualitatively representable atomic algebra then* $\mathbf{A}$ *has a representation with at most* $3|At(\mathbf{A})|^3$ *points in its base.*

- Typically, a lot less than $3|At(\mathbf{A})|^3$.

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Qualitative representations of non-representable RAs and of NAs

Representing the non-representable

Let us play games $\mathcal{G}_n(I_a, \mathbf{A})$ and $\mathcal{QG}_n(I_a, \mathbf{A})$ with $\mathbf{A}$ being the algebra below.

;	=	<	>	#
=	=	<	>	#
<	<	<	1	<, #
>	>	1	>	>, #
#	#	<, #	>, #	=, <, >

- $\mathcal{G}_n(I_a, \mathbf{A})$: player $\forall$ wins.
- $\mathcal{QG}_n(I_a, \mathbf{A})$: player $\exists$ wins!

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$=$	$=$	$<$	$>$	$\#$
$<$	$<$	$<$	1	$<, \#$
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- $\mathcal{G}_n(I_a, \mathbf{A})$: player $\forall$ wins.
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Representing the non-associative

- Consider (again) the algebra $\mathbf{C}_5$ on 6 atoms 1' (identity), a_1 (blue), a_2 (red), a_3 (green), a_4 (yellow), a_5 (black) such that all and only trichromatic triangles are consistent.
- Let us play $\mathcal{QG}_n(I_a, \mathbf{B})$.
- $\exists$ wins, taking hints from this:

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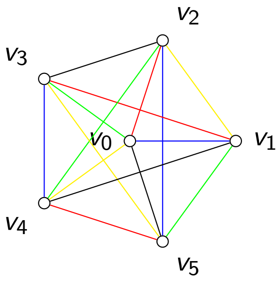

Is everything qualitatively representable?

- Now, consider the algebra $\mathbf{C}_4$ on 5 atoms 1' (identity), a_1 (blue), a_2 (red), a_3 (green), a_4 (yellow) such that all and only trichromatic triangles are consistent.
- Let us play $\mathcal{QG}_n(I_a, \mathbf{C})$.

Theorem (Al-Juaid, Jackson, Koussas, TK)

The algebra $\mathfrak{C}_n$ on $n+1$ atoms is qualitatively representable if and only if n is odd.

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Monk/Ramsey algebras

For $n \geq 2$ let $k(n) = R(\overbrace{3, \ldots, 3}^{n \text{ times}})$, that is, $k(n)$ is a Ramsey number for n-colouring with no monochromatic triangles. Let $\mathbf{M}_n$ be the algebra on the following set of atoms

$$\{1'\} \cup \{a_1^k : k < \frac{k(n)(k(n)-1)}{2}\} \cup \{a_i : 1 < i \leq n\}$$

which are all self-converse. Forbidden triples:

- $(1', x, y)$ for $x \neq y$ (and Peircean transforms of these).
- Triples of atoms with the same subscript, that is, (a_i, a_i, a_i) for $1 < i \leq n$, and $(a_1^k, a_1^{k'}, a_1^{k^*})$ for $k, k', k^* < \frac{k(n)(k(n)-1)}{2}$.
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$\mathbf{M}_n$ has no qualitative representation

Lemma

The algebra $\mathbf{M}_n$ is not qualitatively representable (is not in QRA), for any $n \geq 2$.

Proof.

- Any qualitative representation witnessing all consistent triples of atoms is a complete graph with at least $\frac{k(n)(k(n)-1)}{2}$ distinct edges: one for each atom a_1^k, and one for each a_i with $i > 1$.
- Hence, it has at least $k(n)$ distinct points.
- But, by Ramsey Theorem, there is no n-colouring of a set with this many elements. □

Ultraproduct has a (strong) representation

Now consider a non-principal ultraproduct $\mathbf{M} = \prod_{n\in\omega} \mathbf{M}_n/U$ of the $\mathbf{M}_n$, where U is a non-principal ultrafilter over ω. The atoms of $\mathbf{M}$ are, up to isomorphism:

$$\{1'\} \cup \{a_1^\beta \colon \beta < \kappa\} \cup \{a_\gamma \colon 1 < \gamma < \eta\}$$

where κ, η are infinite ordinals. In more detail:

- $a_1^\beta = \langle s_n \colon n \in \omega\rangle/U$ such that $\{n \colon s_n = a_1^k \text{ for some } k\} \in U$,
- $a_\gamma = \langle s_n \colon n \in \omega\rangle/U$ such that $\{n \colon s_n = a_i \text{ for some } i\} \in U$.

The forbidden triangles are exactly as in $\mathbf{M}_n$. Intuitively, we now have infinitely many colours, one of which comes in infinitely many shades.

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QRA is not finitely based

Lemma

The algebra **M** *is a representable relation algebra.*

Proof.

- We use atomic representability game.
- At each stage s, a strict network N_s is constructed that uses only finitely many atoms.
- So, $\exists$ can satisfy any consistent demand of $\forall$ by extending N_s (if necessary) and completing the network N_{s+1} using as a label an atom not used in N_s. □

Theorem

The variety QRA is not finitely based. Indeed, QRA is not finitely aximatised in FOL.

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Complexity of qualitative representability

Qualitative representability is NP-complete

Theorem (Jackson, Hirsch, TK)

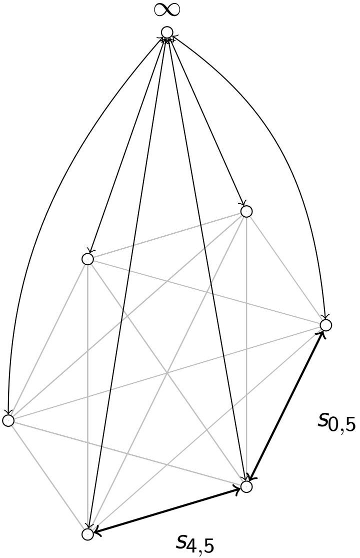
The problem of determining whether a finite NA has a qualitative representation is NP-complete.

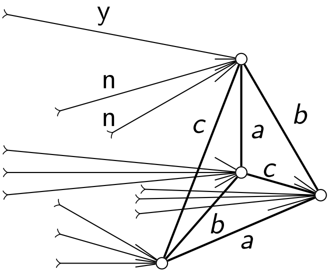
This contrasts with the corresponding question for ordinary representations which Hirsch and Hodkinson proved to be undecidable.

Proof.

- The problem is in NP, because if a qualitative representation exists, then there exists a "small" one checkable in cubic time.
- For NP-hardness, we reduce the 3-colouring problem for finite graphs. More precisely, we encode 4-colouring problem for a certain expansion of any finite graph G. □

Encoding: a picture first





Encoding: basic setup

- A finite digraph $G = (V, \mathcal{E})$ with vertices V and edges $\mathcal{E}$.
- Adjoin three new points $\top$, $*$ and $\circ$ to V such that $\top$ is adjacent to all vertices of V and $*$, $\circ$ are isolated. This is to make sure that:
  - If there is a 4-colouring, then it must use all 4 colours.
  - There is a non-edge between vertices coloured the same.
  - There is a non-edge between vertices coloured differently.
  - There is an independent 3-element set.
- Call the expanded graph $G^\infty = (V^\infty, \mathcal{E}^\infty)$.
- Let g be a symbol not appearing in V^∞. Define a non-associative atom structure $(S, I, \breve{}, C)$ as follows:

$$S = \{1', ab, ac, ad, bc, bd, cd, \mathrm{y}, \mathrm{y}', \mathrm{n}, \mathrm{n}'\} \cup \{s_{uv} : (u, v) \in \mathcal{E}^\infty\} \cup \{g\}$$

  - The identity is an atom: $I = 1'$.
  - Converses are: $s_{uv}\breve{} = s_{vu}$ and $\mathrm{y}\breve{} = \mathrm{y}'$, $\mathrm{n}\breve{} = \mathrm{n}'$; all other atoms are self-converse.

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$$S = \{1', ab, ac, ad, bc, bd, cd, \mathrm{y}, \mathrm{y}', \mathrm{n}, \mathrm{n}'\} \cup \{s_{uv} : (u, v) \in \mathcal{E}^\infty\} \cup \{g\}$$

  - The identity is an atom: $I = 1'$.
  - Converses are: $s_{uv}\breve{} = s_{vu}$ and $\mathrm{y}\breve{} = \mathrm{y}'$, $\mathrm{n}\breve{} = \mathrm{n}'$; all other atoms are self-converse.

Encoding: basic setup

- A finite digraph $G = (V, \mathcal{E})$ with vertices V and edges $\mathcal{E}$.
- Adjoin three new points $\top$, $*$ and $\circ$ to V such that $\top$ is adjacent to all vertices of V and $*$, $\circ$ are isolated. This is to make sure that:
  - If there is a 4-colouring, then it must use all 4 colours.
  - There is a non-edge between vertices coloured the same.
  - There is a non-edge between vertices coloured differently.
  - There is an independent 3-element set.
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Encoding: consistent triangles

All Peircean transforms of the following:

- $(1', x, x)$ for every $x \in S$ (*this says all edges are symmetric*)
- (i, j, k) for $i, j, k \in \{ab, ac, ad, bc, bd, cd\}$ and $|\{i, j, k\}| = 3$ and where i, j, k involve exactly three letters from $\{a, b, c, d\}$ (*this encodes a complete graph for 4-colouring* G^∞).
- (s_{uv}, s_{vw}, s_{uw}) whenever $(u, v), (v, w), (u, w) \in \mathcal{E}^\infty$ (*this encodes all triangles appearing in* G^∞)
- (s_{uv}, s_{vw}, g) when $(u, v), (v, w) \in \mathcal{E}^\infty$ but $(u, w) \notin \mathcal{E}^\infty \cup \{=\}$ (*this encodes all triangles consisting of two edges and one non-edge*)
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- (g, g, g) (*this encodes a triangle of three non edges, which exists because there is an independent 3-element set*)
- $(\alpha, \mathrm{y}', \mathrm{n})$, $(\alpha, \mathrm{n}', \mathrm{y})$ and $(\alpha, \mathrm{n}', \mathrm{n})$ for $\alpha \in \{ab, ac, ad, bc, bd, cd\}$ (*this encodes the colouring mechanism*)
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Fact

The reduction maps $(V^\infty, \mathcal{E}^\infty)$ to the complex algebra of the atom structure $(S, I, \breve{}, C)$.

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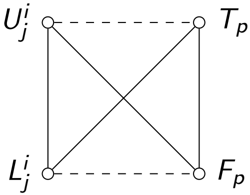
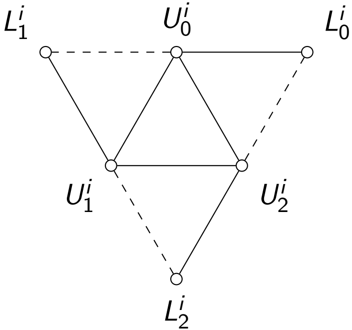
Complexity of qualitative network satisfiability is very often NP-complete

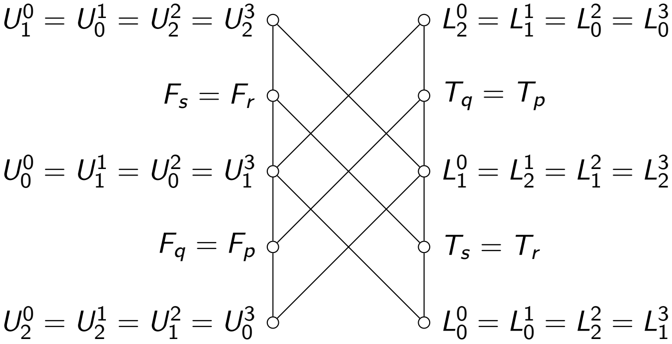
Encoding NAE3SAT

- Let us call a QRA **A** special, if **A** has a relation *a* somewhat resembling $\leq$, and a relation $\#$ somewhat resembling incomparability.
- Given an instance of NAE3SAT, we associate (partial) networks below with a position in a clause and a literal (left), and with a clause (right).





Network satisfiability – often NP-complete



Picture I. Interpretation of an instance of M-NAE-3-SAT: the formula $(p \vee q \vee r) \vee (q \vee r \vee s) \vee (r \vee s \vee p) \vee (s \vee p \vee q)$.

NP completeness

Theorem

Let $\mathbf{A}$ *be a special algebra. If the configuration of Picture I embeds into a qualitative representation of* $\mathbf{A}$*, then network satisfaction problem for any fragment of* $\mathbf{A}$ *containing relations* $\leq$ *and* $\#$ *is NP-complete.*

Thank you!