Theorem Provers Part 2: Inefficient Computation

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Theorem Provers & Reflected Computation

In part 4 of this lecture series, we'll look at proof by reflection: yet another take on proofs as computer programs.

Some relevant ingredients to recall:

- From day 2, problems that can be inefficient to prove:

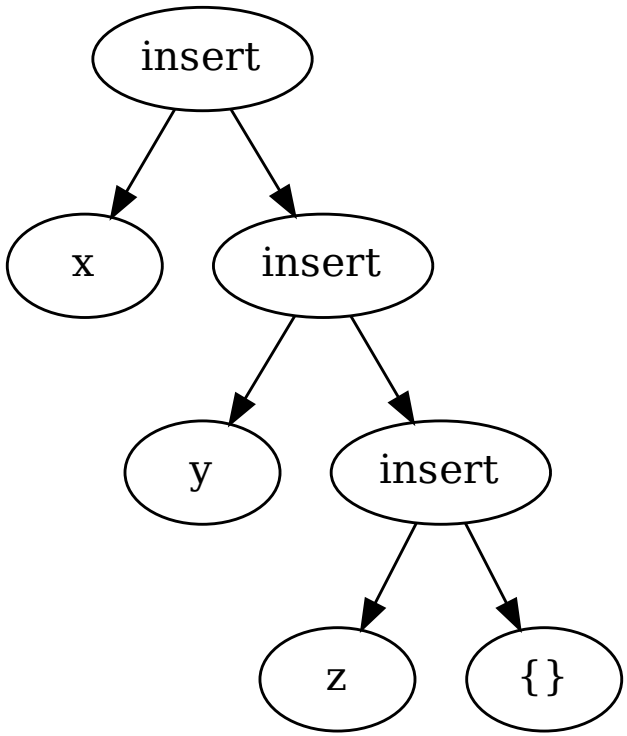
$$\{x, 3, 4, 5, 6\} = \{6, 5, 4, 3, x\}$$

- From day 3, that we aim to validate tedious/exhaustive proofs.
- From day 3 (& week 1), that we can prove things about programs.

Back to Set Equality

To revisit our set equality problem, the issue is that the syntax is linear.

$$\{x, 3, 4, 5, 6\} = \{6, 5, 4, 3, x\}$$



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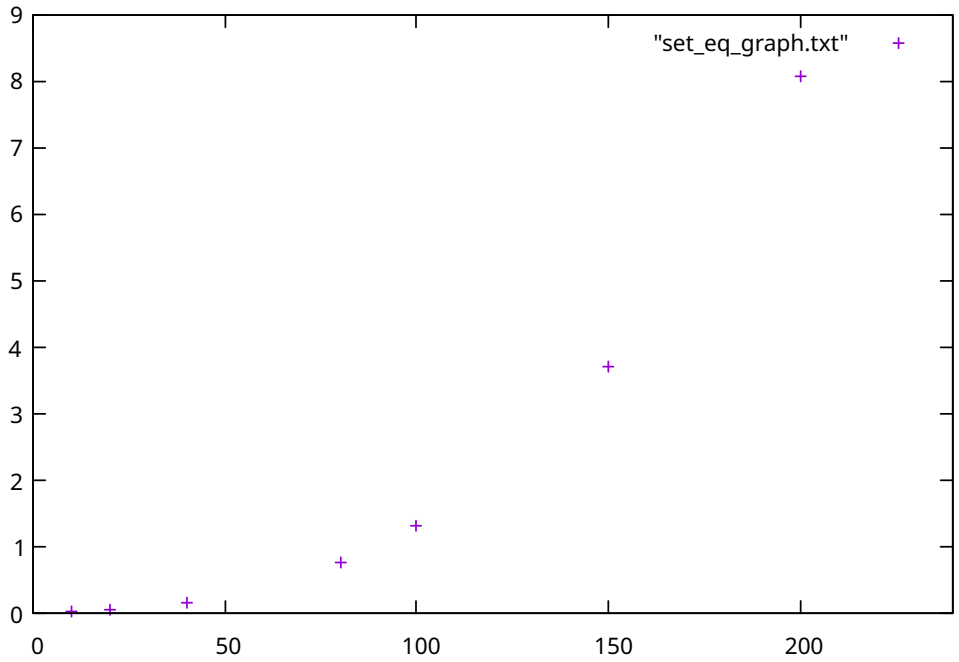
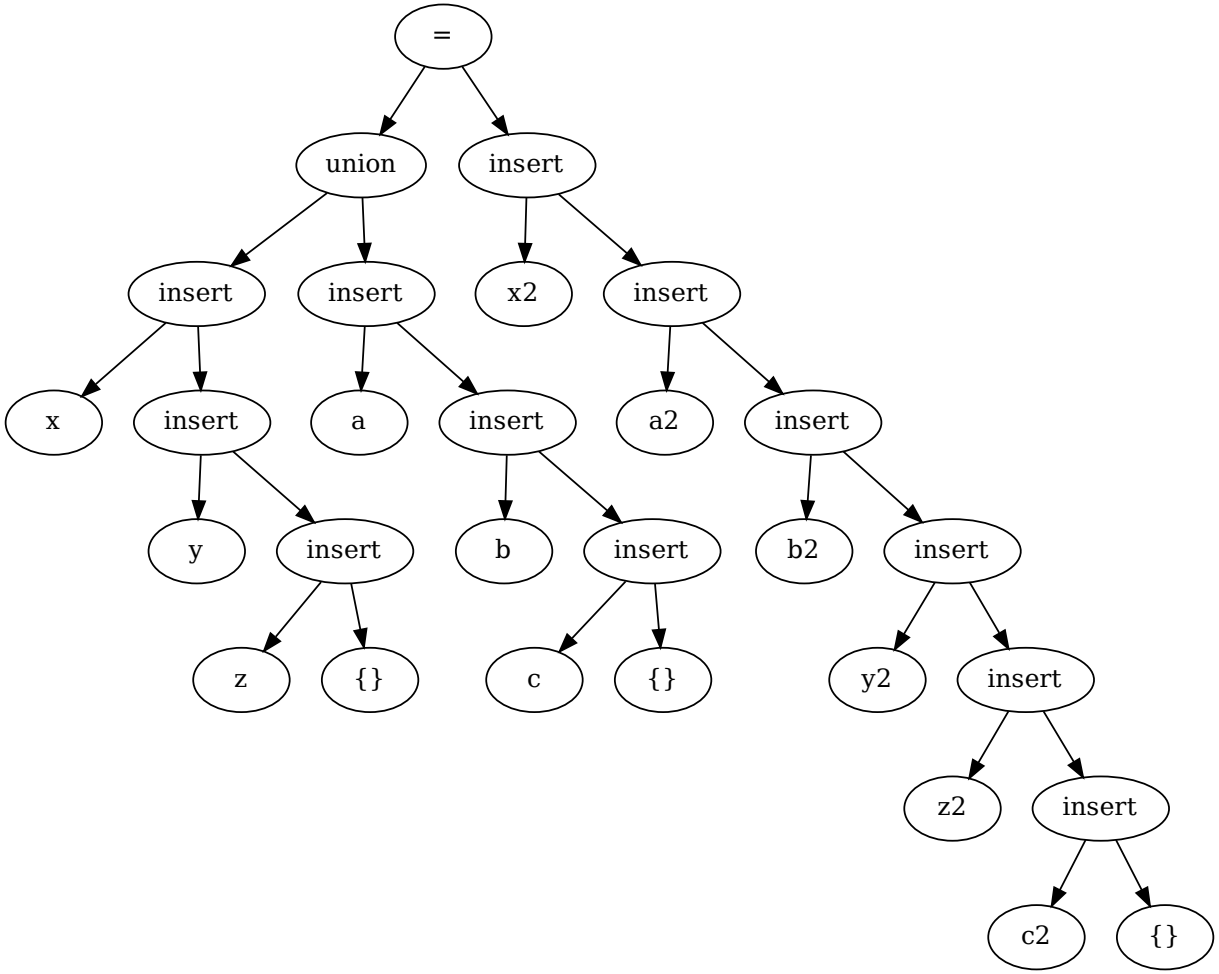


Our graph is (a) quadratic and (b) the numbers are small.

Adding Unions

$$\{x, y, z\} \cup \{a, b, c\} = \{x, a, b, y, z, c\}$$



Sorting via Union Split/Merge

$$\{x, a, b, y, z, c\} = \{x, y, z\} \cup \{a, b, c\}$$

We can write a proof tool based on this strategy. Given a set construction:

1. Sort the syntax (in our implementation language).
2. If the construction is in order, stop.
3. Divide the sorted syntax into “front” half and “back” half.
4. Propose a division into the two halves, and prove it in $O(n)$ steps.
5. Recurse on the halves.

This is about the best we can do in Isabelle/HOL.

Pros and Cons

Strengths of this (kind of) approach:

- We can just do it (in an LCF-style tool).
- Complexity is $\approx O(n \log(n))$ proof steps.

Limitations:

- The algorithm is fiddly and specialised.
- Complexity is $\approx O(n \log(n)^2)$ in the implementation.

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Is this useful in practice? I have seen it most often involving language infrastructure.

Computations in Logic

For **pure, functional, terminating** programs, their meaning is the same in the logic as in computation.

For instance `map ::` $(\alpha$ `->` $\beta)$ `->` α `list ->` β `list`

- `map f [] = []`
- `map f (x : xs) = f x : map f xs`

Not true for impure functions (e.g. `getEnv :: string -> string`) or for potentially nonterminating functions. In these cases, the logical/mathematical type needs to acknowledge the potential effects.

Doing Computation in Logic

Given `f :: (`α `-> `β`)`, `x :: `α, how can we prove that `f x = y`?

We can write an evaluator using ordinary proof tools in the logic.

We could also just evaluate `f x` in a conventional programming language environment.

- Typically orders of magnitude faster.
  - In principle it is a constant factor.
- Adds the language environment to the trusted core.

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Most intuitionistic tools have such an interpreter. Most HOL implementations do not. ACL2 exposes (a safe subset of) the host Lisp implementation.

Appealing to a Logic

We have also seen:

- How to characterise a logic.
- How to prove a logic L_1 sound (in L_2).
- How (in principle) to implement a logic as a program.

So, given a goal G in our host theorem prover

- phrase an equivalent goal G_1 in an inner logic L_1
- run `checker` G_1 and get `True`
- conclude from the correctness of `checker` that $L_1 \vdash G_1$
- from the soundness of L_1 that G

Deep and Shallow Encodings

To use this trick, we need to construct a **deep encoding**.

This is a standard distinction when we encode a logic or language in another. Do we map expressions to their syntax or their semantics?

- Is 1 + 1 the same thing as 2?
- Is $\mathit{True} \wedge \mathit{True}$ the same thing as True?
- Does $\{1, 2, 3\}$ have a first element?

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```
datatype α Set_Encoding = Insert α (α Set_Encoding)
    | Empty

fun interp ::  (α -> β) -> α Set_Encoding -> β Set
    where
        interp f Empty = {}
        interp f (Insert x s) = ({f x} ∪ interp s)
```

Deep and Shallow Encodings and Reflection

$$\{x, 3, 4, 5, 6\} = \{6, 5, 4, 3, x\}$$

```
datatype Number_Encoding = Numeral Int | Variable String
fun interp2 ::  (String -> Int) -> Number_Encoding -> Int
    where
        interp2 env (Numeral i) = i
        interp2 env (Variable nm) = env nm
fun interp3 ::  (String -> Int) ->
                Number_Encoding Set_Encoding -> Int Set
    where
        interp3 env enc = interp (interp2 env) enc
```

```
fun interp3 ::  (String -> Int) ->
                Number_Encoding Set_Encoding -> Int Set
```

To make use of this, we need a gadget that looks at our goal G, and invents an encoding G_{env} and G_{enc} such that `interp3` G_{env} G_{enc} = G.

This is a **reflection** of part of the logic back into itself.

We can then manipulate this deep encoding, and reason about the interpretation of these manipulations.

Reflected Term Normalisation in Coq

$$\{x, 3, 4, 5, 6\} = \{6, 5, 4, 3, x\}$$

I used this puzzle and its relatives to help learn Isabelle, HOL4, ACL2 and Coq.

In Coq:

- Define a type of deep encodings of relevant syntax.
- Map the goal G to a deep reflection of itself G_{enc}.
- Define an ordinary sorting/normalising pass `f`.
- Verify `f`, i.e. show $\forall syn.$ `interp (f` syn`)` $\vdash$ `interp` syn.
- Check that `interp` $G_{enc} = G$.

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- Check that `interp` $G_{enc} = G$.

It's neat that this works! But, the reflection part was disappointing.

Reflected Meta-Strategies in ACL2

$$\{x, 3, 4, 5, 6\} = \{6, 5, 4, 3, x\}$$

In ACL2:

- Every term is in the type of Lisp trees.
  - `('x, 3, 4, 5, 6)` & `(car, cdr)`
- Define a syntactic criterion for relevant syntax.
  - ACL2 synthesises an equivalent of `interp`.
- Define an ordinary Lisp sorting/normalising pass `f`.
- Verify `f`, i.e. show $\forall x$. `f` $x \neq$ `nil --> interp (f` x`) = interp` x.
- Add `f` as a meta-method.

Performance-wise, this works really well! But it's Lisp.

More Reflected Logic?

We've looked at a silly exercise for reflection.

What about running a general hosted logic implementation?

- A tableau solver?
- A SAT solver?
- An implementation of a general-purpose logic?

Recap

Recap. How can we efficiently prove a goal?

- Assume a computational escape-hatch for `f x = y`.
- Reflect goal G into an encoding G_{enc}.
- Run a relevant/efficient proof from some logical prover L.
- Appeal to the soundness of L.

That's all for today. Thanks for listening.

Questions?