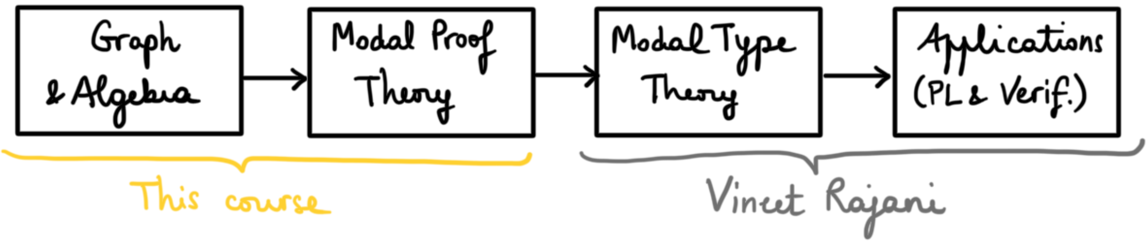
Intuitionistic Modal Logic

Sonia Marin

Lecture 1

What this course is about

Intuitionistic modal logics have seen several waves of interest and are currently a quite active area of research



Monday: Intuitionism & Intuitionistic Logic

Tuesday: Intuitionistic Modal Logic:
- Syntax & Semantics

Wednesday:
- Soundness & Completeness
- Proof System(s)

Thursday: & Proof Search

Friday: Further Topics:
Interesting fragments & Extensions

Intuitionism

School of mathematics founded by **Brouwer** (~1907)

→ rejects mathematical methods whose justification required an abstract concept of "truth"

→ instead, a mathematical object must be given by a construction and a statement can only be true if there is a proof of it

In particular:

- Law of Excluded Middle (lem)

$$A \lor \neg A$$

can only be true if there is an algorithm to decide A

- Double Negation Elimination (dne)

$$\neg\neg A \longrightarrow A$$

The BHK Proof Interpretation

Brouwer-Heyting-Kolmogorov

(1) A proof of $A \land B$ is given by a proof of A and a proof of B

$$\frac{\begin{matrix}\Pi\\ A\end{matrix} \qquad \begin{matrix}\Pi\\ B\end{matrix}}{A \land B}\land_I$$

(2) A proof of $A \lor B$ is given by either a proof of A or a proof of B (regarded as evidence for $A \lor B$)

$$\frac{\begin{matrix}\Pi\\ A\end{matrix}}{A \lor B}\lor_{I1} \qquad \frac{\begin{matrix}\Pi\\ B\end{matrix}}{A \lor B}\lor_{I2}$$

(3) A proof of $A \to B$ is given by a construction that transforms any proof of A into a proof of B

$$\frac{\begin{matrix}[A]\\ \|\\ B\end{matrix}}{A \to B}\to_I$$

(4) There are no proof of $\bot$

Intuitionistic Logic

Heyting (Brouwer's student) developed a logic which axiomatised these ideas formally (~1928)
[Kolmogorov had an earlier version (~1925) which seems to have remained unknown in the West for several years]

Formulas $A ::= p^{\in Prop} \mid A \wedge A \mid A \vee A \mid A \longrightarrow A \mid \bot$

we are considering the propositional fragment

Connectives are independent (no de Morgan duality)

Negation: $\neg A := A \rightarrow \bot$

Axiomatic (Hilbert) system

Implication

$$\begin{cases} A \rightarrow (B \rightarrow A) \\ (A \rightarrow (B \rightarrow C)) \longrightarrow ((A \rightarrow B) \longrightarrow (A \rightarrow C)) \end{cases}$$

these axioms are enough to define classical prop logic with $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ as all other connectives can be defined subsequently

Conjunction

$$\begin{cases} (A \wedge B) \longrightarrow A \\ (A \wedge B) \longrightarrow B \\ A \rightarrow (B \rightarrow (A \wedge B)) \end{cases}$$

Disjunction

$$\begin{cases} A \longrightarrow (A \vee B) \\ B \longrightarrow (A \vee B) \\ (A \rightarrow C) \rightarrow ((B \rightarrow C) \longrightarrow ((A \vee B) \rightarrow C)) \end{cases}$$

False $\bot \rightarrow A$

Deduction System

Let us write Γ for a set of formulas.

$$\dfrac{A \in \Gamma}{\Gamma \vdash A} \qquad \dfrac{A \text{ axiom}}{\Gamma \vdash A} \qquad \dfrac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}\,\text{mp}$$

Deduction Theorem If $\Gamma, A \vdash B$ then $\Gamma \vdash A \to B$

Proof: by induction on $\vdash$

- $\overline{\Gamma, A \vdash A}$ $\Rightarrow$ $\Gamma \vdash A \to A$
  (using the fact that $\cdot \vdash A \to A$)
- $\overline{\Gamma, A \vdash B}$ where B axiom $\Rightarrow$ $\overline{\Gamma \vdash B}$ & $\overline{\Gamma \vdash B \to (A \to B)}$
  $\Rightarrow$ $\Gamma \vdash A \to B$ by mp
- $$\dfrac{\Gamma, A \vdash B \qquad \Gamma, A \vdash B \to C}{\Gamma, A \vdash C}$$

$\Rightarrow$ by IH $\Gamma \vdash A \to B$ & $\Gamma \vdash A \to (B \to C)$

& $\Gamma \vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$\Rightarrow$ $\Gamma \vdash A \to C$ by mp ☺

Exercise

$$\underbrace{A \vee (A \to \bot)}_{\text{lem}} \vdash \underbrace{((A \to \bot) \to \bot) \to A}_{\text{dne}}$$

$\vdash (A \to \text{dne}) \to ((\neg A \to \text{dne}) \to (\text{lem} \to \text{dne}))$ (axiom)

$\vdash A \to \text{dne} = A \to (\neg\neg A \to A)$ (axiom)

$\vdash \neg A \to \text{dne} = (A \to \bot) \to (((A \to \bot) \to \bot) \to A)$

↳ let $\Gamma = \{A \to \bot, (A \to \bot) \to \bot\}$

$$\dfrac{\dfrac{\overline{\Gamma \vdash A \to \bot} \qquad \overline{\Gamma \vdash (A \to \bot) \to \bot}}{\Gamma \vdash \bot} \qquad \overline{\Gamma \vdash \bot \to A}}{\Gamma \vdash A}$$

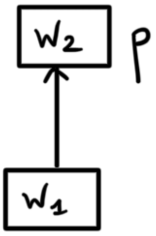
Semantics

One attempt to explain intuitionism in terms of how one acquires knowledge over time is given by Kripke semantics

A **Kripke model** is a tuple $(W, \leqslant, \Vdash)$

- W is a non empty set of "worlds"
- $\leqslant$ is a partial order on W
- $\Vdash \subseteq W \times Prop$ is a monotone valuation
  i.e. if $u \leqslant v$ and $u \Vdash p$ then $v \Vdash p$

↳ can be represented graphically



means $(\{w_1, w_2\}, w_1 \leqslant w_2, w_2 \Vdash p)$

Note: lem : $p \vee \neg p$ at w_1 ?

Exercise Show that $w_1 \nVdash ((A \to B) \to A) \to A$ (Pierce's Law)

We extend the relation $\Vdash$ to $W \times Form$ by induction

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \to B$ iff for all $v \geqslant u$ if $v \Vdash A$ then $v \Vdash B$
- $u \Vdash \bot$ never

Lemma Monotonicity transfers to formulas.

Proof: Exercise

Def: $\vDash A$ if for all model $(W, \leqslant, \Vdash)$ and all $u \in W$: $u \Vdash A$

Theorem : Soundness & Completeness

Kripke semantics is sound & complete for intuitionistic propositional logic, that is :

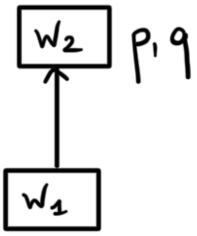
$$\vdash A \quad \text{iff} \quad \vDash A$$

Proof : We will discuss how to prove sound & complete IML and the methods will be applicable to IPL too.

This means that we can use the semantics to prove some ppties of $\vdash$

Exercise Show that $p \to q \nvdash \neg p \vee q$ by exhibiting a Kripke model $(W, \leqslant, \Vdash)$ and $w \in W$ s.t.

$w \Vdash p \to q$ but $w \nVdash \neg p \vee q$



$(\{w_1, w_2\}, w_1 \leqslant w_2, \{w_2 \Vdash p, w_2 \Vdash q\})$

$w_1 \Vdash p \to q$

but $w_1 \nVdash \neg p$ because $w_2 \Vdash p$ and $w_2 \nVdash \bot$

and $w_1 \nVdash q$ so $w_1 \nVdash \neg p \vee q$

Proof Systems

Formally, the best way to characterize intuitionistic logic is by natural deduction system "à la Gentzen"

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array} \qquad\qquad \begin{array}{c}\Pi\\ B\end{array}}{A \wedge B}\,\wedge_I \qquad\qquad \dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\,\wedge E1 \qquad \dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\,\wedge E2$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\,\vee_{I1} \qquad \dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\,\vee_{I2} \qquad\qquad \dfrac{A \vee B \qquad \begin{array}{c}[A]\\ \|\\ C\end{array} \quad \begin{array}{c}[B]\\ \|\\ C\end{array}}{C}\,\vee_E$$

$$\dfrac{\begin{array}{c}[A]\\ \|\\ B\end{array}}{A \to B}\,\to_I \qquad\qquad \dfrac{A \to B \qquad A}{B}\,\to_E$$

$$\dfrac{\bot}{A}\,\bot_E$$

Sequent systems

Keep track explicitly of all assumptions and conclusions at each step of the proof

with syntax $\Gamma \Rightarrow C$

set of assumptions (Γ) conclusion (C)

→ critically replaces modus ponens $\dfrac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}$

by a cut-rule $\dfrac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow B}{\Gamma \Rightarrow B}$ (which will be admissible)

$$\dfrac{}{\Gamma, A \Rightarrow A}\,\text{id} \qquad\qquad \dfrac{}{\Gamma, \bot \Rightarrow C}\,\bot e$$

$$\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\wedge r \qquad\qquad \dfrac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\,\wedge e$$

$$\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,\vee r_1 \qquad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,\vee r_2 \qquad\qquad \dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\,\vee e$$

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\to r \qquad\qquad \dfrac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\,\to e$$

We build proofs inductively from leaves id and $\bot e$ as finite trees using the rules.

We say that $\Gamma \Rightarrow A$ is provable if there is a proof.

Exercise Show $A \vee \neg A \Rightarrow ((A \to B) \to A) \to A$ provable.

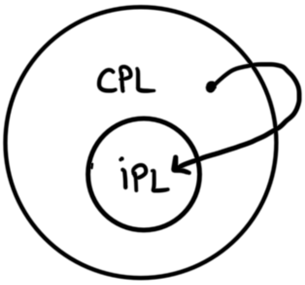
$$\dfrac{\dfrac{}{A, (A \to B) \to A \Rightarrow A}\,\text{id} \qquad \dfrac{\dfrac{\dfrac{\text{id}\,\dfrac{}{A \Rightarrow A} \qquad \bot e\,\dfrac{}{\bot, A \Rightarrow B}}{A \to \bot, A \Rightarrow B}}{A \to \bot \Rightarrow A \to B} \qquad \text{id}\,\dfrac{}{A \to \bot, A \Rightarrow A}}{A \to \bot, (A \to B) \to A \Rightarrow A}}{\dfrac{A \vee \neg A, ((A \to B) \to A) \Rightarrow A}{A \vee \neg A \Rightarrow ((A \to B) \to A) \to A}}$$

Theorem (Soundness & Completeness)

$\Gamma \vdash A$ iff $\Gamma \Rightarrow A$ provable.

Proof: Again, the tools discussed in the rest of the course will allow to conclude here

Translating classical into intuitionistic



Intuitionistic logic is "weaker" than classical logic as a fragment

But it is strong enough to emulate all classical reasoning

Negative translation:

$$
\begin{aligned}
p^N &:= \neg\neg p \\
(A \wedge B)^N &:= A^N \wedge B^N \\
(A \vee B)^N &:= \neg(\neg A^N \wedge \neg B^N) \\
(A \to B)^N &:= A^N \to B^N \\
\bot^N &:= \bot
\end{aligned}
$$

Theorem A is provable in CPL iff A^N is provable in IPL.

Proof: Can be proved via sequent system or via semantics.