

Intuitionistic Modal Logic

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Lecture 2

Modal Logic - Recap

Formulas $A ::= p \in \text{Prop} \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \perp \mid \Box A \mid \Diamond A$

We have two new primitives: the modalities $\begin{cases} \text{necessity } \Box \\ \text{possibility } \Diamond \end{cases}$

Axiomatisation

$\text{CPL} \begin{cases} A \rightarrow (B \rightarrow A) \\ (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\ (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A) \end{cases}$ *can be substituted with modal formulas*

+ $k : \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

+ duality : $\Diamond A \leftrightarrow \neg \Box \neg A$

Deduction System

Let us write Γ for a set of formulas.

$$\frac{A \in \Gamma}{\Gamma \vdash^K A}$$

$$\frac{A \text{ axiom}}{\Gamma \vdash^K A}$$

$$\frac{\Gamma \vdash^K A \quad \Gamma \vdash^K A \rightarrow B}{\Gamma \vdash^K B} \text{ mp}$$

Necessitation :

$$\frac{\Gamma \vdash^K A}{\Gamma \vdash^K \Box A} \text{ nec}$$

Logic K is the set of formulas s.t. $\vdash^K A$.

Deduction Theorem $\Gamma, A \vdash^K B$ then $\Gamma \vdash^K A \rightarrow B$

Proof: Same as before

• $\frac{\vdash^K B}{\Gamma, A \vdash^K \Box B}$ nec need to show $\Gamma \vdash^K A \rightarrow \Box B$

by necessitation we also get $\Gamma \vdash^K \Box B$

then apply mp to axiom $\Gamma \vdash \Box B \rightarrow (A \rightarrow \Box B)$ ☺

Exercise Show that $\Box(A \rightarrow \perp) \vee \Box B \vdash^K \Box(A \rightarrow B)$

$$\frac{\frac{B \rightarrow (A \rightarrow B)}{\Box(B \rightarrow (A \rightarrow B))} \text{ nec}}{\Box B \rightarrow \Box(A \rightarrow B)} K$$

$$\frac{\frac{A \vdash A \quad A \rightarrow \perp \vdash A \rightarrow \perp}{A \rightarrow \perp, A \vdash \perp} \quad \frac{\vdash \perp \rightarrow B}{\perp \vdash B}}{\frac{A \rightarrow \perp, A \vdash B}{(A \rightarrow \perp) \rightarrow (A \rightarrow B)} \text{ nec} + K} \Box(A \rightarrow \perp) \vdash \Box(A \rightarrow B)$$

Semantics A Kripke modal model is a structure $(W, R, \Vdash)$

- W is a non-empty set of "worlds"
- R is a generic binary relation on W
 $\hookrightarrow$ called accessibility/visibility relation
- $\Vdash$ is a generic valuation relation on $W \times \text{Prop}$
 $\hookrightarrow$ in fact functional so can write $V: W \rightarrow \mathcal{P}(\text{Prop})$
 $V(u) = \{p \in \text{Prop} \mid u \Vdash p\}$

The satisfaction relation is given by:

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \rightarrow B$ iff if $u \Vdash A$ then $u \Vdash B$ $\leftarrow$
- $u \Vdash \perp$ never
- $u \Vdash \Box A$ iff for all $v \in W$: if $u R v$ then $v \Vdash A$
- $u \Vdash \Diamond A$ iff there exists $v \in W$: $u R v$ and $v \Vdash A$

A is valid if for all model $\underbrace{(W, R, \Vdash)}_{\mathcal{M}}$ & for all $u \in W$: $u \Vdash A$
 $\models A$ $\mathcal{M} \models A$

Theorem

A is a theorem of K iff A is valid

$$\vdash^K A$$

$$\models A$$

Exercise Show that $M \models \Box p \rightarrow p$ iff M is reflexive

$$\Box p \rightarrow \Box \Box p$$

transitive



$$u \Vdash \Box p \Rightarrow u \Vdash p$$



$$u \Vdash \Box p \Rightarrow w \Vdash p \Rightarrow u \Vdash \Box \Box p$$

Standard Translation

Classical modal logic arises from interpreting the above definitions of modal model in the meta-theory of classical mathematics

First-order language L_m of modal models :

- one binary symbol R
- a unary symbol P for each $p \in \text{Prop}$
- no functions, no constants

For variable x and modal formula A :

$$\left. \begin{array}{l} p^x = P(x) \\ \perp^x = \perp \\ (A \wedge B)^x = A^x \wedge B^x \\ (A \vee B)^x = A^x \vee B^x \\ (A \rightarrow B)^x = A^x \rightarrow B^x \\ (\Box A)^x = \forall y (x R y \rightarrow A^y) \\ (\Diamond A)^x = \exists y (x R y \wedge A^y) \end{array} \right\} \begin{array}{l} \text{codes up the possible world} \\ \text{conditions for } A \text{ satisfied at } x \end{array}$$

Theorem

$$K \vdash A \quad \text{iff} \quad \text{FOL} \vdash \forall x A^x$$

Alex Simpson's IML

PhD from Univ of Edinburgh in 1994

Requirement 6 There is an intuitionistically comprehensible explanation of the modalities, relative to which IML is sound and complete.

Proposal: $IK \vdash A$ iff $IFOL \vdash \forall x A^x$

The intuitionistic element is confined to a formal theory (namely intuitionistic first-order logic) which lets it be independent from the informal meta-theory being used (hence may as well be classical as in maths)

Gisèle Fischer-Servi's IML

[1977 - 84]

Goal: to axiomatise intuitionistic modal logics as reasonable analogues of the classical modal systems

Axiomatisation

$IPL \left\{ \begin{array}{l} A \rightarrow (B \rightarrow A) \\ (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\ \text{other axioms} \end{array} \right.$ can be substituted with modal formulas

+ $k_{\Box}$: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

+ duality: $\Diamond A \leftrightarrow \neg \Box \neg A$

↳ what connection between the two modalities?

Gödel translation

from IPL to modal logic.

$$\begin{aligned}\perp^* &:= \perp \\ p^* &:= \Box p \quad \text{for } p \in \text{Prop} \\ (A \wedge B)^* &:= A^* \wedge B^* \\ (A \vee B)^* &:= A^* \vee B^* \\ (A \rightarrow B)^* &:= \Box(A^* \rightarrow B^*)\end{aligned}$$

Theorem (McKinsey-Tarski)

$$\text{IPL} \vdash A \quad \text{iff} \quad \text{S4} \vdash A^*$$

$$\begin{aligned}K &+ t: \Box A \rightarrow A \\ &+ 4: \Box A \rightarrow \Box \Box A\end{aligned}$$

or equivalently, the modal logic of reflexive
and transitive structures

Proposal: translation from IK to $\text{S4} \times K$

bimodal logic $\left\{ \begin{array}{l} \Box_1 \text{ an S4 modality} \\ \Box_2 \text{ a K-modality} \end{array} \right.$

By developing this approach, she arrives to the following possible formulation of IK:

$$\begin{aligned}\Diamond(A \vee B) &\rightarrow (\Diamond A \vee \Diamond B) \\ (\Box A \wedge \Box B) &\rightarrow \Box(A \wedge B) \\ \Diamond \perp &\rightarrow \perp \\ \Diamond(A \rightarrow B) &\rightarrow (\Box A \rightarrow \Diamond B) \\ (\Diamond A \rightarrow \Box B) &\rightarrow \Box(A \rightarrow B)\end{aligned}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \rightarrow B}{\Gamma \vdash B}$$

$$\frac{\cdot \vdash A \rightarrow B}{\Gamma \vdash \Box A \rightarrow \Box B}$$

$$\frac{\cdot \vdash A \rightarrow B}{\Gamma \vdash \Diamond A \rightarrow \Diamond B}$$

Semantics

[Fischer-Servi / Plotkin & Stirling]

It is natural to find a way to combine semantics for intuitionistic prop logic and for modal logic.

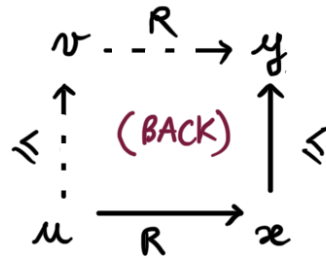
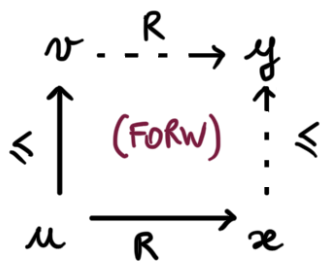
Birelational model $(W, \leq, R, \Vdash)$

- W non empty set of "worlds"
- $\leq$ preorder on W
- R binary accessibility relation on W
- $\Vdash$ monotone valuation: if $u \leq v$ and $u \Vdash p$ then $v \Vdash p$

together with confluence conditions

(FORW) If $u \leq v$ and $u R x$
then there exists $y \in W$ s.t. $v R y$ and $x \leq y$

(BACK) If $u R x$ and $x \leq y$
then there exists $v \in W$ s.t. $u \leq v$ and $v R y$



We extend the relation $\Vdash$ to $W \times \text{Form}$ by induction

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \rightarrow B$ iff for all $v \geq u$ if $v \Vdash A$ then $v \Vdash B$
- $u \Vdash \perp$ never
- $u \Vdash \Box A$ iff for all $v \geq u$ for all w s.t. $v R w$: $w \Vdash A$
- $u \Vdash \Diamond A$ iff there exists v s.t. $u R v$: $v \Vdash A$

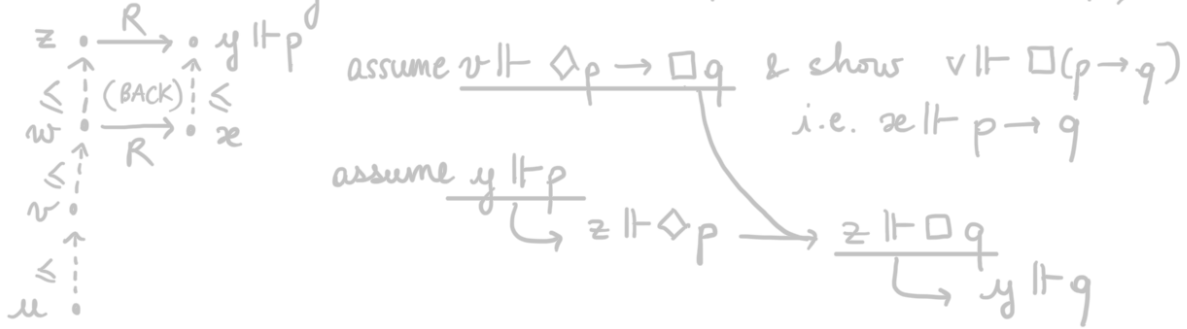
intuitionistic part

modal part

Exercise

Show that $\models (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$

Take arbitrary birelational model $(W, \leq, R, \Vdash)$ and $u \in W \neq \emptyset$.



Lemma Monotonicity transfers to formulas.

Proof: By induction on formulas

• For $\Diamond$: Show that if $u \leq v$ and $u \Vdash \Diamond A$ then $v \Vdash \Diamond A$
 $u \Vdash \Diamond A$ means there is x s.t. $u R x$ and $x \Vdash A$



Plotkin & Stirling's IML

[1986]

Proposal: Define IML as an axiomatisation of validity relative to arbitrary BIRELATIONAL structure

Axiomatisation

$\text{IPL} \left\{ \begin{array}{l} A \rightarrow (B \rightarrow A) \\ (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\ \text{other axioms} \end{array} \right.$ can be substituted with modal formulas

+ $k_{\Box}$: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

+ duality: $\Diamond A \leftrightarrow \neg \Box \neg A$

$k_{\Diamond}$: $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

$n_{\Diamond}$: $\Diamond \perp \rightarrow \perp$

$c_{\Diamond}$: $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$

$i_{\Diamond\Box}$: $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$

Deduction System

Let us write Γ for a set of formulas.

$$\frac{A \in \Gamma}{\Gamma \vdash^K A}$$

$$\frac{A \text{ axiom}}{\Gamma \vdash^K A}$$

$$\frac{\Gamma \vdash^K A \quad \Gamma \vdash^K A \rightarrow B}{\Gamma \vdash^K B} \text{ mp}$$

Necessitation:

$$\frac{\vdash^K A}{\Gamma \vdash^K \Box A} \text{ nec}$$

Note that there are no additional rules for $\Diamond$ in this version so the Deduction Theorem holds directly.

Logic IK

axiomatised as Fischer-Servi
or Plotkin & Stirling

Theorem The three proposals are equivalent

$$IK \vdash A \quad \text{iff} \quad \models A \quad \text{iff} \quad IFOL \vdash \forall x A^x$$

Soundness by induction on $\vdash^K$

• nec $\frac{\vdash^K A}{\Gamma \vdash^K \Box A}$ suppose $u \Vdash \Gamma$ and show that $u \Vdash \Box A$
take $v \triangleright u$ and x s.t. $v R x$ then $x \Vdash A$ by IH

Completeness

By contrapositive assume $IK \not\vdash A$

The proof aims at building a countermodel $(W, R, \leq, \Vdash)$
with $u \in W$ s.t. $u \not\Vdash A$.

Proof Method

A canonical model is a universal counter-model
— it invalidates all unprovable formulas.

By the Prime Lemma
 as $\nVdash^K A$, there is Γ prime such that $\Gamma \nVdash^K A$
 hence (in particular) $A \notin \Gamma$

By the Truth Lemma
 we have that in $\mathcal{M}_c$: $\Gamma \nVdash A$ thus $\nVdash A$.

- (1) A set of formulas Γ is prime
- if $\Gamma \vdash^K A$ then $A \in \Gamma$ (deductively closed)
 - $\Gamma \nVdash^K \perp$ (consistent)
 - if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$ (disjunction property)

(2) Prime Lemma

If $\Gamma \vdash^K A$ then there is a prime set $\Delta \supseteq \Gamma$ s.t. $\Delta \vdash^K A$

Proof: Similar to the Lindenbaum Lemma
 that any consistent theory can be extended to
 a maximally consistent one (in FOL)
 — for which the standard proof uses Zorn Lemma

(3) The canonical model $\mathcal{M}_c$ = $(W, \leq, R, \Vdash)$ with

- $W := \{\Gamma \text{ prime}\}$
- $\leq := \subseteq$
- $\Gamma R \Delta$ iff $\{\Diamond A \mid A \in \Delta\} \subseteq \Gamma$ and $\{B \mid \Box B \in \Gamma\} \subseteq \Delta$
- $\Gamma \Vdash p$ iff $p \in \Gamma$

Need to show that $\mathcal{M}_c$ is indeed a birelational model

(4) Truth Lemma

in $\mathcal{M}_c$: $\Gamma \Vdash A$ iff $A \in \Gamma$

Proof: by induction on A