Intuitionistic Modal Logic

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Lecture 2

Modal Logic – Recap

Formulas $A ::= p^{\in Prop} \mid A \wedge A \mid A \vee A \mid A \to A \mid \bot \mid \Box A \mid \Diamond A$

We have two new primitives: the modalities $\begin{cases} \text{necessity} & \Box \\ \text{possibility} & \Diamond \end{cases}$

Axiomatisation

can be substituted with modal formulas

$$\text{CPL} \begin{cases} A \to (B \to A) \\ (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (A \to B) \to (\neg B \to \neg A) \end{cases}$$

$+ \quad k : \quad \Box(A \to B) \to (\Box A \to \Box B)$

$+ \quad$ duality $: \quad \Diamond A \leftrightarrow \neg \Box \neg A$

Deduction System

Let us write Γ for a set of formulas.

$$\frac{A \in \Gamma}{\Gamma \vdash^{K} A} \qquad \frac{A \text{ axiom}}{\Gamma \vdash^{K} A} \qquad \frac{\Gamma \vdash^{K} A \qquad \Gamma \vdash^{K} A \to B}{\Gamma \vdash^{K} B}\,mp$$

Necessitation: $\quad \dfrac{\cdot \vdash^{K} A}{\Gamma \vdash^{K} \Box A}\,nec$

Logic K is the set of formulas s.t. $\vdash^{K} A$.

Deduction Theorem $\quad \Gamma, A \vdash^{K} B$ then $\Gamma \vdash^{K} A \to B$

Proof: Same as before

- $$\frac{\cdot \vdash^{K} B}{\Gamma, A \vdash^{K} \Box B}\,\text{nec}$$ need to show $\Gamma \vdash^{K} A \to \Box B$

by necessitation we also get $\Gamma \vdash^{K} \Box B$
then apply mp to axiom $\Gamma \vdash \Box B \to (A \to \Box B)$ ☺

Exercise Show that $\quad \Box(A \to \bot) \vee \Box B \vdash^{K} \Box(A \to B)$

$$\cfrac{\cfrac{B \to (A \to B)}{\Box(B \to (A \to B))}\,\text{nec}}{\Box B \to \Box(A \to B)}\,k$$

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A \qquad A \to \bot \vdash A \to \bot}{A \to \bot, A \vdash \bot} \qquad \cfrac{\vdash \bot \to B}{\bot \vdash B}}{A \to \bot, A \vdash B}}{(A \to \bot) \to (A \to B)}}{\Box(A \to \bot) \vdash \Box(A \to B)}\,\text{nec}+k$$

Semantics A Kripke modal model is a structure $(W, R, \Vdash)$
- W is a non-empty set of "worlds"
- R is a generic binary relation on W
  ↪ called accessibility / visibility relation
- $\Vdash$ is a generic valuation relation on $W \times \text{Prop}$
  ↪ in fact functional so can write $V : W \to \mathcal{P}(\text{Prop})$
  $V(u) = \{ p \in \text{Prop} \mid u \Vdash p \}$

The satisfaction relation is given by:
- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \to B$ iff if $u \Vdash A$ then $u \Vdash B$ ←
- $u \Vdash \bot$ never
- $u \Vdash \Box A$ iff for all $v \in W$: if uRv then $v \Vdash A$
- $u \Vdash \Diamond A$ iff there exists $v \in W$: uRv and $v \Vdash A$

A is valid if for all model $\underbrace{(W, R, \Vdash)}_{\mathcal{M}}$ & $\underbrace{\text{for all } u \in W : u \Vdash A}_{\mathcal{M} \vDash A}$
$\vDash A$

Theorem

A is a theorem of K iff A is valid

$$\vdash^{K} A \qquad\qquad \vDash A$$

Exercise Show that $\mathcal{M} \vDash \Box p \to p$ iff $\mathcal{M}$ is reflexive

$\Box p \to \Box\Box p$ transitive

$$\begin{tikzcd} u \arrow[loop above] \end{tikzcd}$$

$u \Vdash \Box p \implies u \Vdash p$

$$\begin{tikzcd} u \arrow[r] \arrow[rr, bend left] & v \arrow[r] & w \end{tikzcd}$$

$u \Vdash \Box p \implies w \Vdash p \implies u \Vdash \Box\Box p$

Standard Translation

Classical modal logic arises from interpreting the above definitions of modal model in the meta-theory of classical mathematics

First-order language $\mathcal{L}_{m}$ of modal models :

- one binary symbol R
- a unary symbol P for each $p \in Prop$
- no functions, no constants

For variable x and modal formula A :

$$\left.\begin{aligned}
p^{x} &= P(x)\\
\perp^{x} &= \perp\\
(A \wedge B)^{x} &= A^{x} \wedge B^{x}\\
(A \vee B)^{x} &= A^{x} \vee B^{x}\\
(A \to B)^{x} &= A^{x} \to B^{x}\\
(\Box A)^{x} &= \forall y\ (xRy \to A^{y})\\
(\Diamond A)^{x} &= \exists y\ (xRy \wedge A^{y})
\end{aligned}\right\}$$

codes up the possible world conditions for A satisfied at x

Theorem $K \vdash A$ iff $FOL \vdash \forall x\, A^{x}$

Alex Simpson's IML

PhD from Univ of Edinburgh in 1994

Requirement 6 There is an intuitionistically comprehensible explanation of the modalities, relative to which IML is sound and complete.

Proposal: $IK \vdash A$ iff $IFOL \vdash \forall x\, A^x$

The intuitionistic element is confined to a formal theory (namely intuitionistic first-order logic) which lets it be independent from the informal meta-theory being used (hence may as well be classical as in maths)

Gisèle Fischer-Servi's IML

[1977 – 84]

Goal: to axiomatise intuitionistic modal logics as reasonable analogues of the classical modal systems

Axiomatisation

can be substituted with modal formulas

$$\text{I}\,\text{CPL} \begin{cases} A \to (B \to A) \\ (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ \cancel{(\neg A \to \neg B) \to (B \to A)} \quad \text{other axioms} \end{cases}$$

$+ \quad k_{\Box}: \quad \Box(A \to B) \to (\Box A \to \Box B)$

~~+ duality: $\Diamond A \leftrightarrow \neg\Box\neg A$~~

↳ what connection between the two modalities?

Gödel translation from IPL to modal logic.

$$\bot^* := \bot$$
$$p^* := \Box p \quad \text{for } p \in Prop$$
$$(A \wedge B)^* := A^* \wedge B^*$$
$$(A \vee B)^* := A^* \vee B^*$$
$$(A \to B)^* := \Box(A^* \to B^*)$$

Theorem (McKinsey-Tarski)

$$IPL \vdash A \quad \text{iff} \quad S4 \vdash A^*$$

$$\boxed{\begin{array}{l} K + t: \Box A \to A \\ \quad + 4: \Box A \to \Box\Box A \end{array}}$$

or equivalently, the modal logic of reflexive and transitive structures

Proposal: translation from IK to $\underbrace{S_4 \times K}$

bimodal logic $\begin{cases} \Box_1 \text{ an S4 modality} \\ \Box_2 \text{ a K-modality} \end{cases}$

By developing this approach, she arrives to the following possible formulation of IK:

$$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$$
$$(\Box A \wedge \Box B) \to \Box(A \wedge B)$$
$$\Diamond\bot \to \bot$$
$$\Diamond(A \to B) \to (\Box A \to \Diamond B)$$
$$(\Diamond A \to \Box B) \to \Box(A \to B)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B} \qquad \frac{\cdot \vdash A \to B}{\Gamma \vdash \Box A \to \Box B} \qquad \frac{\cdot \vdash A \to B}{\Gamma \vdash \Diamond A \to \Diamond B}$$

Semantics

[Fischer-Servi / Plotkin & Stirling]

It is natural to find a way to <u>combine semantics</u> for intuitionistic prop logic and for modal logic.

<u>Birelational model</u> $(W, \leqslant, R, \Vdash)$

- W non empty set of "worlds"
- $\leqslant$ <u>preorder</u> on W
- R binary <u>accessibility</u> relation on W
- $\Vdash$ <u>monotone valuation</u>: if $u \leqslant v$ and $u \Vdash p$ then $v \Vdash p$

together with <u>confluence conditions</u>

(FORW) If $u \leqslant v$ and uRx
then there exists $y \in W$ s.t. vRy and $x \leqslant y$

(BACK) If uRx and $x \leqslant y$
then there exists $v \in W$ s.t. $u \leqslant v$ and vRy

$$\begin{tikzcd}
v \arrow[r, dashed, "R"] & y \\
u \arrow[u, "\leqslant"] \arrow[r, "R"'] \arrow[ur, phantom, "\text{(FORW)}"] & x \arrow[u, dotted, "\leqslant"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
v \arrow[r, dashed, "R"] & y \\
u \arrow[u, dotted, "\leqslant"] \arrow[r, "R"'] \arrow[ur, phantom, "\text{(BACK)}"] & x \arrow[u, "\leqslant"']
\end{tikzcd}$$

We extend the relation <u>$\Vdash$ to $W \times Form$</u> by induction

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \to B$ iff for all $v \geqslant u$ if $v \Vdash A$ then $v \Vdash B$
- $u \Vdash \bot$ never
- $u \Vdash \Box A$ iff for all $v \geqslant u$ for all w s.t vRw : $w \Vdash A$
- $u \Vdash \Diamond A$ iff there exists v s.t. uRv : $v \Vdash A$

intuitionistic part

modal part

Exercise Show that $\vDash (\Diamond p \to \Box q) \to \Box(p \to q)$

Take arbitrary birelational model $(W, \leqslant, R, \Vdash)$ and $u \in W \neq \emptyset$.

$$\begin{tikzcd} z \arrow[r, "R"] & y \Vdash p \\ w \arrow[u, dashed, "\leqslant"] \arrow[r, "R"] \arrow[ur, phantom, "(BACK)"] & x \arrow[u, dashed, "\leqslant"'] \\ v \arrow[u, dashed, "\leqslant"] & \\ u \arrow[u, dashed, "\leqslant"] & \end{tikzcd}$$

assume $\underline{v \Vdash \Diamond p \to \Box q}$ & show $v \Vdash \Box(p \to q)$
i.e. $x \Vdash p \to q$

assume $\underline{y \Vdash p}$
$\hookrightarrow z \Vdash \Diamond p \longrightarrow \underline{z \Vdash \Box q}$
$\hookrightarrow y \Vdash q$

Lemma Monotonicity transfers to formulas.

Proof: By induction on formulas

- For $\Diamond$: Show that if $u \leqslant v$ and $u \Vdash \Diamond A$ then $v \Vdash \Diamond A$

$u \Vdash \Diamond A$ means there is x s.t uRx and $x \Vdash A$

$$\begin{tikzcd} v \arrow[r, dashed] & y \\ u \arrow[u] \arrow[r] \arrow[ur, phantom, "(FORW)"] & x \arrow[u, dashed] \end{tikzcd}$$

by IH $y \Vdash A$ hence $v \Vdash \Diamond A$

Plotkin & Stirling's IML

[1986]

Proposal: Define IML as an axiomatisation of validity relative to arbitrary BIRELATIONAL structure

Axiomatisation

can be substituted with modal formulas

ICPL
$$\begin{cases} A \to (B \to A) \\ (A \to (B \to C)) \longrightarrow ((A \to B) \longrightarrow (A \to C)) \\ \cancel{(A \to B) \to (\neg B \to \neg A)} \quad \text{other axioms} \end{cases}$$

$+ \quad k_\Box: \quad \Box(A \to B) \longrightarrow (\Box A \longrightarrow \Box B)$

~~+ duality : $\Diamond A \longleftrightarrow \neg\Box\neg A$~~

$k_\Diamond: \quad \Box(A \to B) \longrightarrow (\Diamond A \to \Diamond B)$

$n_\Diamond: \quad \Diamond \bot \to \bot$

$c_\Diamond: \quad \Diamond(A \vee B) \longrightarrow (\Diamond A \vee \Diamond B)$

$i_{\Diamond\Box}: \quad (\Diamond A \to \Box B) \longrightarrow \Box(A \to B)$

Deduction System

Let us write Γ for a set of formulas.

$$\frac{A \in \Gamma}{\Gamma \vdash^{IK} A} \qquad \frac{A \text{ axiom}}{\Gamma \vdash^{IK} A} \qquad \frac{\Gamma \vdash^{IK} A \qquad \Gamma \vdash^{IK} A \to B}{\Gamma \vdash^{IK} B}\ mp$$

Necessitation:

$$\frac{\cdot \vdash^{IK} A}{\Gamma \vdash^{IK} \Box A}\ nec$$

Note that there are no additional rules for $\Diamond$ in this version so the Deduction Theorem holds directly.

axiomatised as Fischer-Servi or Plotkin & Stirling

$\boxed{\text{Logic IK}}$

Theorem The three proposals are equivalent

$$IK \vdash A \quad \text{iff} \quad \vDash A \quad \text{iff} \quad IFOL \vdash \forall x A^x$$

Soundness by induction on $\vdash^{IK}$

- nec $\dfrac{\cdot \vdash^{IK} A}{\Gamma \vdash^{IK} \Box A}$ suppose $u \Vdash \Gamma$ and show that $u \Vdash \Box A$
  take $v \geq u$ and x s.t vRx then $x \Vdash A$ by IH

Completeness

By contrapositive assume $IK \nvdash A$
The proof aims at building a countermodel $(W, R, \leq, \Vdash)$ with $u \in W$ s.t. $u \nVdash A$.

Proof Method

A canonical model is a universal counter-model — it invalidates all unprovable formulas.

By the Prime Lemma
as $\not\vdash^{IK} A$, there is Γ prime such that $\Gamma \not\vdash^{IK} A$
hence (in particular) $A \notin \Gamma$

By the Truth Lemma
we have that in $\mathcal{M}_c$: $\Gamma \not\Vdash A$ thus $\not\models A$.

(1) A set of formulas Γ is prime
- if $\Gamma \vdash^{IK} A$ then $A \in \Gamma$ (deductively closed)
- $\Gamma \not\vdash^{IK} \bot$ (consistent)
- if $A \vee B \in \Gamma$ then $A \in \Gamma$ or $B \in \Gamma$ (disjunction property)

(2) Prime Lemma
If $\Gamma \not\vdash^{IK} A$ then there is a prime set $\Delta \supseteq \Gamma$ s.t. $\Delta \not\vdash^{IK} A$

Proof: Similar to the Lindenbaum Lemma
that any consistent theory can be extended to
a maximally consistent one (in FOL)
— for which the standard proof uses Zorn Lemma

(3) The canonical model $\mathcal{M}_c = (W, \leqslant, R, \Vdash)$ with
- $W := \{\Gamma \text{ prime}\}$
- $\leqslant := \subseteq$
- $\Gamma R \Delta$ iff $\{\Diamond A \mid A \in \Delta\} \subseteq \Gamma$ and $\{B \mid \Box B \in \Gamma\} \subseteq \Delta$
- $\Gamma \Vdash p$ iff $p \in \Gamma$

Need to show that $\mathcal{M}_c$ is indeed a birelational model

(4) Truth Lemma
in $\mathcal{M}_c$: $\Gamma \Vdash A$ iff $A \in \Gamma$

Proof: by induction on A