Intuitionistic Modal Logic

Sonia Marin

Lecture 3

Logic IK – Recap

Syntax

IPL +
$k_\Box$ $\Box(A\to B)\to(\Box A\to\Box B)$
$k_\Diamond$: $\Box(A\to B)\to(\Diamond A\to\Diamond B)$
$n_\Diamond$: $\Diamond\bot\to\bot$
$c_\Diamond$: $\Diamond(A\vee B)\to(\Diamond A\vee\Diamond B)$
$i_{\Diamond\Box}$: $(\Diamond A\to\Box B)\to\Box(A\to B)$

$$\frac{A\in\Gamma}{\Gamma\vdash^{IK}A}\qquad\frac{A \text{ axiom}}{\Gamma\vdash^{IK}A}\qquad\frac{\Gamma\vdash^{IK}A\qquad\Gamma\vdash^{IK}A\to B}{\Gamma\vdash^{IK}B}\,mp\qquad\frac{\cdot\vdash^{IK}A}{\Gamma\vdash^{IK}\Box A}\,nec$$

Standard Translation

For variable x and modal formula A:

$$\begin{aligned}
p^x &= P(x)\\
\bot^x &= \bot\\
(A\wedge B)^x &= A^x\wedge B^x\\
(A\vee B)^x &= A^x\vee B^x\\
(A\to B)^x &= A^x\to B^x\\
(\Box A)^x &= \forall y\,(xRy\to A^y)\\
(\Diamond A)^x &= \exists y\,(xRy\wedge A^y)
\end{aligned}$$

Semantics

Birelational model $(W, \leqslant, R, \Vdash)$

- W non empty set of "worlds"
- $\leqslant$ preorder on W
- R binary accessibility relation on W
- $\Vdash$ monotone valuation : if $u \leqslant v$ and $u \Vdash p$ then $v \Vdash p$

together with confluence conditions

(FORW) If $u \leqslant v$ and uRx
then there exists $y \in W$ s.t. vRy and $x \leqslant y$

$$\begin{tikzcd} v \arrow[r, dashed] & y \\ u \arrow[u] \arrow[r] & x \arrow[u, dashed] \end{tikzcd}$$

(BACK) If uRx and $x \leqslant y$
then there exists $v \in W$ s.t. $u \leqslant v$ and vRy

$$\begin{tikzcd} v \arrow[r, dashed] & y \\ u \arrow[u, dashed] \arrow[r] & x \arrow[u] \end{tikzcd}$$

defining satisfaction at world u as:

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \to B$ iff for all $v \geqslant u$ if $v \Vdash A$ then $v \Vdash B$
- $u \Vdash \bot$ never
- $u \Vdash \Box A$ iff for all $v \geqslant u$ for all w s.t. vRw : $w \Vdash A$
- $u \Vdash \Diamond A$ iff there exists v s.t. uRv : $v \Vdash A$

Theorem $IK \vdash A$ iff $\models A$ iff $IFOL \vdash \forall x\, A^x$

From axiomatic systems to structural proof theory

Graph & Algebra → Modal Proof Theory → Modal Type Theory → Applications (PL & Verif.)

Soundness & Completeness

Normalisation & Proof Search

Sequent Calculus for IPL

$$\dfrac{}{\Gamma, A \Rightarrow A}\,\text{id} \qquad\qquad \dfrac{}{\Gamma, \bot \Rightarrow C}\,\bot\ell$$

$$\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\wedge r \qquad\qquad \dfrac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\,\wedge\ell$$

$$\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,\vee r_1 \quad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,\vee r_2 \qquad\qquad \dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\,\vee\ell$$

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\to r \qquad\qquad \dfrac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\,\to\ell$$

Theorem $\Gamma \vdash^{IPL} A$ iff $\Gamma \Rightarrow A$ provable

We want to extend this theorem to $\Gamma \vdash^{IK} A$

This requires in particular to be able to prove <u>all axioms of IK</u>.

given: $\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\to r$ & $\dfrac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\,\to\ell$

we can try:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B, A \Rightarrow A}\,\text{id} \qquad \dfrac{}{B, A \Rightarrow B}\,\text{id}}{A \to B, A \Rightarrow B}\,\to\ell}{\square(A \to B), \square A \Rightarrow \square B}\,\square\ell + \square r}{\square(A \to B) \Rightarrow \square A \to \square B}\,\to r}{k_\square : \square(A \to B) \to (\square A \to \square B)}\,\to r$$

or even more optimistically:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A \Rightarrow A$}
\RightLabel{$\Diamond r$}
\UnaryInfC{$A \Rightarrow \Diamond A$}
\RightLabel{$\vee r_1$}
\UnaryInfC{$A \Rightarrow \Diamond A \vee \Diamond B$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$B \Rightarrow B$}
\RightLabel{$\Diamond r$}
\UnaryInfC{$B \Rightarrow \Diamond B$}
\RightLabel{$\vee r_2$}
\UnaryInfC{$B \Rightarrow \Diamond A \vee \Diamond B$}
\RightLabel{$\vee l$}
\BinaryInfC{$A \vee B \Rightarrow \Diamond A \vee \Diamond B$}
\RightLabel{$\Diamond l$}
\UnaryInfC{$\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B$}
\RightLabel{$\to r$}
\UnaryInfC{$C_\Diamond : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$\Diamond A \to \Box B, A \Rightarrow A$}
\RightLabel{$\Diamond r$}
\UnaryInfC{$\Diamond A \to \Box B, A \Rightarrow \Diamond A$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$B, A \Rightarrow B$}
\RightLabel{$\Box l$}
\UnaryInfC{$\Box B, A \Rightarrow B$}
\RightLabel{$\to l$}
\BinaryInfC{$\Diamond A \to \Box B, A \Rightarrow B$}
\RightLabel{$\to r$}
\UnaryInfC{$\Diamond A \to \Box B \Rightarrow A \to B$}
\RightLabel{$\Box r$}
\UnaryInfC{$\Diamond A \to \Box B \Rightarrow \Box(A \to B)$}
\UnaryInfC{$(\Diamond A \to \Box B) \to \Box(A \to B)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$A \to B, A \Rightarrow A$}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$B, A \Rightarrow B$}
\RightLabel{$\to l$}
\BinaryInfC{$A \to B, A \Rightarrow B$}
\RightLabel{$\Box l + \Diamond l + \Diamond r$}
\UnaryInfC{$\Box(A \to B), \Diamond A \Rightarrow \Diamond B$}
\RightLabel{$\to r$}
\UnaryInfC{$\Box(A \to B) \Rightarrow \Diamond A \to \Diamond B$}
\RightLabel{$\to r$}
\UnaryInfC{$\Box(A \to B) \to (\Diamond A \to \Diamond B)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{$\bot l$}
\UnaryInfC{$\bot \Rightarrow \bot$}
\RightLabel{$\Diamond l$}
\UnaryInfC{$\Diamond \bot \Rightarrow \bot$}
\RightLabel{$\to r$}
\UnaryInfC{$\Diamond \bot \to \bot$}
\end{prooftree}
$$

Theorem $IK \vdash A$ iff $\models A$ iff $IFOL \vdash \forall x\, A^x$

$$
\boxed{
\begin{aligned}
p^x &= P(x) \\
\bot^x &= \bot \\
(A \wedge B)^x &= A^x \wedge B^x \\
(A \vee B)^x &= A^x \vee B^x \\
(A \to B)^x &= A^x \to B^x \\
(\Box A)^x &= \forall y\, (xRy \to A^y) \\
(\Diamond A)^x &= \exists y\, (xRy \wedge A^y)
\end{aligned}
}
$$

Simpson's Sequent Calculus

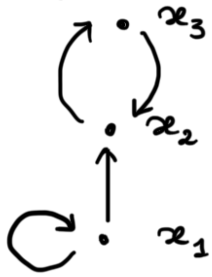
Labelled Sequents

- labels = constants from countable set $\{x, y, z \dots\}$
- labelled formula = $x : A$ ⟵ modal formula
- relational atom = $x R y$
- labelled sequent =

$$\underbrace{\mathcal{R}}_{\text{set of relational atoms}} \mid \underbrace{\Gamma}_{\text{multiset of labelled formulas}} \Rightarrow z : C$$

set of relational atoms → can be interpreted as a directed graph

$\{x_1 R x_2, x_2 R x_3, x_3 R x_2, x_1 R x_1\}$



Rules

$\boxed{\ell IK}$

All the rules for IPL are essentially the same as above (but labelled)

$$\frac{\mathcal{R} \mid \Gamma, z : A \Rightarrow z : B}{\mathcal{R} \mid \Gamma \Rightarrow z : A \to B} \to_r$$

$$\frac{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow x : A \qquad \mathcal{R} \mid \Gamma, x : B \Rightarrow z : C}{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow z : C} \to_\ell$$

The relational context is going to be used for modal rules

$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R} \mid \Gamma \Rightarrow x : \Box A} \Box_r^{*}$$

$$\frac{\mathcal{R}, xRy \mid \Gamma, \overset{x : \Box A}{y : A} \Rightarrow z : C}{\mathcal{R}, xRy \mid \Gamma, x : \Box A \Rightarrow z : C} \Box_\ell$$

$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R}, xRy \mid \Gamma \Rightarrow x : \Diamond A} \Diamond_r$$

$$\frac{\mathcal{R}, xRy \mid \Gamma, \overset{x : \Diamond A}{y : A} \Rightarrow z : C}{\mathcal{R} \mid \Gamma, x : \Diamond A \Rightarrow z : C} \Diamond_\ell^{*}$$

* y is free i.e. y does not appear in the conclusion

Exercise Show that the axioms of IK are provable

$$\dfrac{\dfrac{}{A\to B, A \Rightarrow A}\,\text{id} \qquad \dfrac{}{B, A \Rightarrow B}\,\text{id}}{A\to B,\ A \Rightarrow B}\,\to_\ell$$

$$\boxed{}$$

$$\dfrac{\dfrac{\Box(A\to B),\ \Box A \Rightarrow \Box B}{\Box(A\to B) \Rightarrow \Box A \to \Box B}\,\to_r}{\Box(A\to B) \to (\Box A \to \Box B)}\,\to_r$$

$$\dfrac{}{A \Rightarrow A}\,\text{id} \qquad\qquad \dfrac{}{B \Rightarrow B}\,\text{id}$$

$$\boxed{} \qquad\qquad \boxed{}$$

$$\dfrac{\dfrac{A \Rightarrow \Diamond A}{A \Rightarrow \Diamond A \vee \Diamond B}\,\vee_{r_1} \qquad \dfrac{B \Rightarrow \Diamond B}{B \Rightarrow \Diamond A \vee \Diamond B}\,\vee_{r_2}}{A \vee B \Rightarrow \Diamond A \vee \Diamond B}\,\vee_\ell$$

$$\boxed{}$$

$$\dfrac{\Diamond(A\vee B) \Rightarrow \Diamond A \vee \Diamond B}{c_\Diamond :\ \Diamond(A\vee B) \to (\Diamond A \vee \Diamond B)}\,\to_r$$

Remark In the proof of $\Rightarrow x : A$, all sequents are of the shape

$$\mathcal{R} \mid \Gamma \Rightarrow z : C$$

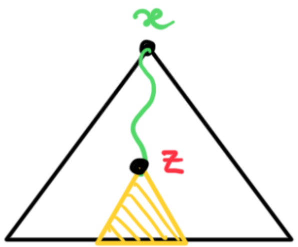
where $\mathcal{R}$ describes an oriented tree rooted at x

↓

a directed acyclic graph whose underlying undirected graph is a tree

i.e. connected acyclic

which implies every pair of vertices is connected by exactly one path



Formula Interpretation

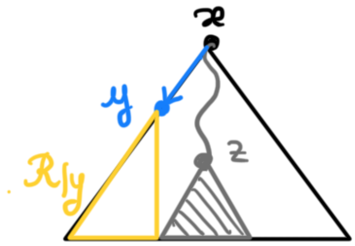
Recall : in IPL $\Gamma \Rightarrow C$ interpreted as $\bigwedge_{A\in\Gamma} A \longrightarrow C$

Let $\mathcal{R} \mid \Gamma \Rightarrow z : C$ with $\mathcal{R}$ is an oriented tree rooted at x

(1) $\lambda^x(\mathcal{R} \mid \Gamma)$

$$= \bigwedge_{x:A\in\Gamma} A \wedge \bigwedge_{xRy\in\mathcal{R}} \Diamond \lambda^y(\mathcal{R}_{|y} \mid \Gamma)$$

↑ subtree rooted at y



(2)

- if $x = z$: $\rho^x(\mathcal{R} \mid \Gamma \Rightarrow z : C)$

$$= \lambda^x(\mathcal{R} \mid \Gamma) \longrightarrow C$$

- otherwise : let u be such that $xRu \in \mathcal{R}$ and $u \rightsquigarrow z$

$$\rho^x(\mathcal{R} \mid \Gamma \Rightarrow z : C)$$

$$= \lambda^x(\mathcal{R} \setminus (xRu, \mathcal{R}_{|u}) \mid \Gamma)$$

$$\longrightarrow \Box \rho^u(\mathcal{R}_{|u} \mid \Gamma \Rightarrow z : C)$$

Example $\underbrace{x_0Rx_1, x_1Rx_2, x_0Rx_3}_{\mathcal{R}} \mid \underbrace{x_2 : A, x_3 : B}_{\Gamma} \Rightarrow x_1 : C$

$$\rho^{x_0}(\mathcal{R} \mid \Gamma \Rightarrow x_1 : C) = \Diamond B \longrightarrow \Box(\Diamond A \longrightarrow C)$$

$$\rho^{x_0}(\mathcal{R} \mid \Gamma \Rightarrow x_1 : C)$$

$$= \lambda^{x_0}(\mathcal{R} \setminus x_0Rx_1 \mid \Gamma) \longrightarrow \Box \rho^{x_1}(\overset{\mathcal{R}\setminus x_0}{\overset{\shortparallel}{x_1Rx_2}} \mid \Gamma \Rightarrow x_1 : C)$$

$$\lambda^{x_0}(x_1Rx_2, x_0Rx_3 \mid \Gamma) \qquad \lambda^{x_1}(x_1Rx_2 \mid \Gamma) \longrightarrow C$$

$$= \Diamond \lambda^{x_3}(x_1Rx_2 \mid \Gamma) \qquad = \Diamond \lambda^{x_2}(\varnothing \mid \Gamma)$$

$$= \Diamond B \qquad = \Diamond A$$

$\boxed{\text{Soundness and Completeness}}$

Theorem $\quad \vdash^{IK} A \quad$ iff $\quad \Rightarrow x : A$ is provable in ℓIK
(for any x)

$$\begin{tikzcd}
\ell IK \vdash \Rightarrow x:A \arrow[r, "\text{① soundness}"] & IK \vdash A \arrow[r, leftrightarrow] \arrow[dl, "\text{② simulation}"] & \vDash A \\
\ell IK + \text{cut} \vdash \Rightarrow x:A \arrow[u, "\text{③ cut admissibility}"] & &
\end{tikzcd}$$

in the background

$$\frac{\mathcal{R} \mid \Gamma \Rightarrow y:B \qquad \mathcal{R} \mid \Gamma, y:B \Rightarrow z:C}{\mathcal{R} \mid \Gamma \Rightarrow y:C}$$

① soundness

for each rule of ℓIK, $IK \vdash \bigwedge \rho(\text{prem.}) \to \rho(\text{concl.})$

$$\Box_r \frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y:A}{\mathcal{R} \mid \Gamma \Rightarrow x:\Box A}$$

$$\rho^x(\mathcal{R}, xRy \mid \Gamma \Rightarrow y:A) = \lambda^x(\mathcal{R} \mid \Gamma) \to \Box \underbrace{\rho^y(\mathcal{R} \mid \Gamma \Rightarrow y:A)}_{(\lambda^y(\mathcal{R} \mid \Gamma) \to A)}$$

$\lambda^y(\mathcal{R} \mid \Gamma)$ ↳ empty conjunction $= \top$

$$\rho^x(\mathcal{R} \mid \Gamma \Rightarrow x:\Box A) = \lambda^x(\mathcal{R} \mid \Gamma) \to \Box A$$

② simulation

each axiom and rule of IK can be simulated by ℓIK using the cut rule

$$\frac{\Gamma \vdash^{IK} A \qquad \Gamma \vdash^{IK} A \to B}{\Gamma \vdash^{IK} B}\ mp$$

$$\frac{\mathcal{R} \mid \Gamma \Rightarrow x:A \qquad \mathcal{R} \mid \Gamma, x:A \Rightarrow x:B}{\mathcal{R} \mid \Gamma \Rightarrow x:B}\ \text{cut}$$

③ cut admissibility

If $\mathcal{R} \mid \Gamma \Rightarrow x : A$ and $\mathcal{R} \mid \Gamma, x : A \Rightarrow y : B$ are provable in lIK via derivations π_1 & π_2

then so is $\mathcal{R} \mid \Gamma \Rightarrow y : B$.

Proof by induction on $(\,\text{size}(A), \text{height}(\pi_1) + \text{height}(\pi_2))$