Intuitionistic Modal Logic

Sonia Marin

Lecture 4

Labelled Sequent Calculus ℓIK – Recap

Rules for propositional connectives : same but labelled

$$\frac{\mathcal{R} \mid \Gamma, z : A \Rightarrow z : B}{\mathcal{R} \mid \Gamma \Rightarrow z : A \to B} \to_r$$

$$\frac{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow x : A \qquad \mathcal{R} \mid \Gamma, x : B \Rightarrow z : C}{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow z : C} \to_\ell$$

Modal rules : make use of the labels & relational atoms

$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R} \mid \Gamma \Rightarrow x : \Box A} \Box_r^*$$

$$\frac{\mathcal{R}, xRy \mid \Gamma, \overset{x : \Box A}{y : A} \Rightarrow z : C}{\mathcal{R}, xRy \mid \Gamma, x : \Box A \Rightarrow z : C} \Box_\ell$$

$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R}, xRy \mid \Gamma \Rightarrow x : \Diamond A} \Diamond_r$$

$$\frac{\mathcal{R}, xRy \mid \Gamma, \overset{x : \Diamond A}{y : A} \Rightarrow z : C}{\mathcal{R} \mid \Gamma, x : \Diamond A \Rightarrow z : C} \Diamond_\ell^*$$

* y is free i.e. y does not appear in the conclusion

Formula interpretation

We can read each labelled sequent $\mathcal{R} \mid \Gamma \Rightarrow z\, C$ as a modal formula when $\mathcal{R}$ describes an oriented tree.

$\rho^{x}(\mathcal{R} \mid \Gamma \Rightarrow z:C)$

$$= \Big(\bigwedge_{x:A\in\Gamma} A \wedge \bigwedge_{\substack{xRy\in\mathcal{R} \\ y\neq u}} \Diamond \lambda^{y}(\mathcal{R}|_{y} \mid \Gamma) \Big) \longrightarrow \Box \rho^{u}(\mathcal{R}|_{u} \mid \Gamma \Rightarrow z:C)$$

Note: Michael suggested to simplify the definition by replacing $\mathcal{R}|_y$ and $\mathcal{R}|_u$ with simply the full $\mathcal{R}$.
Indeed λ and ρ are themselves annotated by label y or u so they will only explore the subtree of $\mathcal{R}$ rooted at y or u

Soundness and Completeness

Theorem $\vdash^{IK} A$ iff $\Rightarrow x:A$ is provable in ℓIK (for any x)

$$\ell IK \vdash\, \Rightarrow x:A \xrightarrow{\text{① soundness}} IK \vdash A \longleftrightarrow \models A$$

in the background

② simulation: $IK \vdash A \longrightarrow \ell IK + \mathrm{cut} \vdash\, \Rightarrow x:A$

③ cut admissibility: $\ell IK + \mathrm{cut} \vdash\, \Rightarrow x:A \longrightarrow \ell IK \vdash\, \Rightarrow x:A$

$$\boxed{\dfrac{\mathcal{R} \mid \Gamma \Rightarrow y:B \qquad \mathcal{R} \mid \Gamma, y:B \Rightarrow z:C}{\mathcal{R} \mid \Gamma \Rightarrow z:C}\ \mathrm{cut}}$$

① soundness

for each rule of ℓIK, $IK \vdash \bigwedge \rho(\text{prem.}) \to \rho(\text{concl.})$

$$\Box_r\ \dfrac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R} \mid \Gamma \Rightarrow x : \Box A}$$

$$\rho^x(\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A) = \lambda^x(\mathcal{R} \mid \Gamma) \to \Box \rho^y(\mathcal{R} \mid \Gamma \Rightarrow y : A)$$

$$\rho^y(\mathcal{R} \mid \Gamma \Rightarrow y : A) = (\underbrace{\lambda^y(\mathcal{R} \mid \Gamma)}_{\text{empty conjunction} = \top} \to A)$$

$$\rho^x(\mathcal{R} \mid \Gamma \Rightarrow x : \Box A) = \lambda^x(\mathcal{R} \mid \Gamma) \to \Box A$$

② simulation

each axiom and rule of IK can be simulated by ℓIK using the cut rule

$$\dfrac{\Gamma \vdash^{IK} A \qquad \Gamma \vdash^{IK} A \to C}{\Gamma \vdash^{IK} C}\ mp$$

$$\dfrac{\mathcal{R} \mid \Gamma \Rightarrow x : A \qquad \mathcal{R} \mid \Gamma, x : A \Rightarrow z : C}{\mathcal{R} \mid \Gamma \Rightarrow z : C}\ cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy \mid y : A \to B, y : A \Rightarrow y : A}\ id \qquad \dfrac{}{xRy \mid y : B, y : A \Rightarrow y : B}\ id}{xRy \mid y : A \to B,\ y : A \Rightarrow y : B}\ \to_\ell}{xRy \mid x : \Box(A \to B), x : \Box A \Rightarrow y : B}\ \Box_\ell}{x : \Box(A \to B), x : \Box A \Rightarrow x : \Box B}\ \Box_r}{\Rightarrow x : \Box(A \to B) \to (\Box A \to \Box B)}\ \to_r$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xRy \mid y : A \Rightarrow y : A}\ id}{xRy \mid y : A \Rightarrow x : \Diamond A}\ \Diamond_r}{xRy \mid y : A \Rightarrow x : \Diamond A \vee \Diamond B}\ \vee_{r_1} \qquad \dfrac{\dfrac{\dfrac{}{xRy \mid y : B \Rightarrow y : B}\ id}{xRy \mid y : B \Rightarrow x : \Diamond B}\ \Diamond_r}{xRy \mid y : B \Rightarrow x : \Diamond A \vee \Diamond B}\ \vee_{r_2}}{xRy \mid y : A \vee B \Rightarrow x : \Diamond A \vee \Diamond B}\ \vee_\ell}{x : \Diamond(A \vee B) \Rightarrow x : \Diamond A \vee \Diamond B}\ \Diamond_\ell}{\Rightarrow x : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)}\ \to_r$$

③ cut admissibility

Lemma

If $\overset{\pi_1}{\mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A}}$ and $\overset{\pi_2}{\mathcal{R} \mid \Gamma, \boxed{x : A} \Rightarrow z : C}$ are provable in ℓIK

then so is $\mathcal{R} \mid \Gamma \Rightarrow z : C$

Proof: by lexicographic induction on $(\text{size}(A), \text{height}(\pi_1) + \text{height}(\pi_2))$

Case analysis:

(1) Key cases: if the last rules in π_1 and π_2 apply to $x : A$

(2) Non-key cases: otherwise

Example for (1):

$$\dfrac{\overset{\pi_1'}{\mathcal{R} \mid \Gamma, x : A \Rightarrow x : B}}{\mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A \to B}} \to_r \qquad \dfrac{\overset{\pi_2'}{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow x : A} \quad \overset{\pi_2''}{\mathcal{R} \mid \Gamma, x : B \Rightarrow z : C}}{\mathcal{R} \mid \Gamma, \boxed{x : A \to B} \Rightarrow z : C} \to_\ell$$

By IH applied to $\pi_1 \vdash \mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A \to B}$
and $\pi_2' \vdash \mathcal{R} \mid \Gamma, \boxed{x : A \to B} \Rightarrow x : A$

$ht(\pi_1) + ht(\pi_2') < ht(\pi_1) + ht(\pi_2)$

we get a proof $\pi' \vdash \mathcal{R} \mid \Gamma \Rightarrow x : A$

By IH applied to $\pi_1' \vdash \mathcal{R} \mid \Gamma, \boxed{x : A} \Rightarrow x : B$
and $\pi' \vdash \mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A}$

$\text{size}(A) < \text{size}(A \to B)$

we get a proof $\pi'' \vdash \mathcal{R} \mid \Gamma \Rightarrow x : B$

By IH applied to $\pi'' \vdash \mathcal{R} \mid \Gamma \Rightarrow \boxed{x : B}$
and $\pi_2'' \vdash \mathcal{R} \mid \Gamma, \boxed{x : B} \Rightarrow z : C$

$\text{size}(B) < \text{size}(A \to B)$

we get a proof $\pi''' \vdash \mathcal{R} \mid \Gamma \Rightarrow z : C$

Example for (2) :

$$\dfrac{\pi_1}{\mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A}} \qquad\qquad \dfrac{\dfrac{\pi_2'}{\mathcal{R} \mid \Gamma, x : A, z : B \Rightarrow z : C}}{\mathcal{R} \mid \Gamma, \boxed{x : A} \Rightarrow z : B \to C} \to_r$$

By IH applied to $\pi_1 \vdash \mathcal{R} \mid \Gamma \Rightarrow \boxed{x : A}$

and $\pi_2' \vdash \mathcal{R} \mid \Gamma, \boxed{x : A}\, z : B \Rightarrow z : C$

$\hookrightarrow$ $ht(\pi_1) + ht(\pi_2') < ht(\pi_1) + ht(\pi_2)$

we get a proof $\pi \vdash \mathcal{R} \mid \Gamma, z : B \Rightarrow z : C$

and therefore we can construct

$$\dfrac{\dfrac{\pi}{\mathcal{R} \mid \Gamma, z : B \Rightarrow z : C}}{\mathcal{R} \mid \Gamma \Rightarrow z : B \to C} \to_r$$

☺

Note

This proof can be extended into a full cut-elimination argument

$$\begin{tikzcd}
 & IK \vdash A \arrow[dl, "2"'] \arrow[rr, leftrightarrow] & & \models A \arrow[dd, "5"', shift right] \\
lIK + cut \vdash \Rightarrow x : A \arrow[dr, "3"'] & & & \\
 & lIK \vdash \Rightarrow x : A \arrow[uu, "1"'] & & flIK \vdash \Rightarrow x : A \arrow[uu, "4"', shift right]
\end{tikzcd}$$

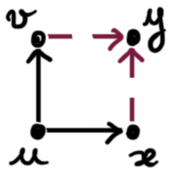
Semantics – Recap

<u>Birelational model</u> $(W, \leqslant, R, \Vdash)$

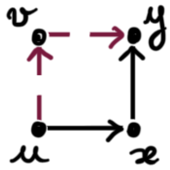
- W non empty set of "worlds"
- $\leqslant$ <u>preorder</u> on W
- R binary <u>accessibility</u> relation on W
- $\Vdash$ <u>monotone valuation</u>: if $u \leqslant v$ and $u \Vdash p$ then $v \Vdash p$

together with <u>confluence conditions</u>

(FORW) If $u \leqslant v$ and uRx
then there exists $y \in W$ s.t. vRy and $x \leqslant y$



(BACK) If uRx and $x \leqslant y$
then there exists $v \in W$ s.t. $u \leqslant v$ and vRy



defining <u>satisfaction at world u</u> as:

- $u \Vdash A \wedge B$ iff $u \Vdash A$ and $u \Vdash B$
- $u \Vdash A \vee B$ iff $u \Vdash A$ or $u \Vdash B$
- $u \Vdash A \to B$ iff for all $v \geqslant u$ if $v \Vdash A$ then $v \Vdash B$
- $u \Vdash \bot$ never
- $u \Vdash \Box A$ iff for all $v \geqslant u$ for all w s.t. vRw : $w \Vdash A$
- $u \Vdash \Diamond A$ iff there exists v s.t. uRv : $v \Vdash A$

Fully Labelled Sequent Calculus

Fully Labelled Sequents

- <u>labels</u> = constants from countable set $\{x, y, z \ldots\}$
- <u>labelled formula</u> = $x : A$ ⟵ modal formula
- <u>relational atoms</u> = xRy $\boxed{x \leqslant y}$
- <u>labelled sequent</u> =

$$\mathcal{R} \mid \Gamma \Rightarrow \Delta$$

<u>set of relational atoms</u> <u>multiset</u> of labelled formulas.

$\boxed{f\ell IK}$

Rules for propositional connectives :

ℓIK:
$$\frac{\mathcal{R} \mid \Gamma \Rightarrow z : A}{\mathcal{R} \mid \Gamma \Rightarrow z : A \vee B} \vee_r$$

$f\ell IK$:
$$\frac{\mathcal{R} \mid \Gamma \Rightarrow \overbrace{z : A, z : B}^{\text{no need to choose a disjunct}}, \Delta}{\mathcal{R} \mid \Gamma \Rightarrow z : A \vee B, \Delta} \vee_r$$

ℓIK:
$$\frac{\mathcal{R} \mid \Gamma, z : A \Rightarrow z : B}{\mathcal{R} \mid \Gamma \Rightarrow z : A \to B} \to_r$$

$f\ell IK$:
$$\frac{\mathcal{R}, z \leq u \mid \Gamma, u : A \Rightarrow u : B, \Delta}{\mathcal{R} \mid \Gamma \Rightarrow z : A \to B, \Delta} \to_r^{*}$$ u fresh

ℓIK:
$$\frac{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow x : A \qquad \mathcal{R} \mid \Gamma, x : B \Rightarrow z : C}{\mathcal{R} \mid \Gamma, x : A \to B \Rightarrow z : C} \to_\ell$$

$f\ell IK$:

the consequent is not deleted!

$$\frac{\mathcal{R}, x \leq y \mid \Gamma, x : A \to B \Rightarrow y : A, z : C, \Delta \qquad \mathcal{R}, x \leq y \mid \Gamma, y : B \Rightarrow z : C, \Delta}{\mathcal{R}, x \leq y \mid \Gamma, x : A \to B \Rightarrow z : C, \Delta} \to_\ell$$

Modal rules :

ℓIK :
$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R} \mid \Gamma \Rightarrow x : \Box A} \Box_r^{*}$$ y is free

$f\ell IK$:
$$\frac{\mathcal{R}, x \leq y, yRz \mid \Gamma \Rightarrow z : A, \Delta}{\mathcal{R} \mid \Gamma \Rightarrow x : \Box A, \Delta} \Box_r^{*}$$ y, z are free

ℓIK:
$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A}{\mathcal{R}, xRy \mid \Gamma \Rightarrow x : \Diamond A} \Diamond_r$$

$f\ell IK$:
$$\frac{\mathcal{R}, xRy \mid \Gamma \Rightarrow y : A, \Delta}{\mathcal{R}, xRy \mid \Gamma \Rightarrow x : \Diamond A, \Delta} \Diamond_r$$

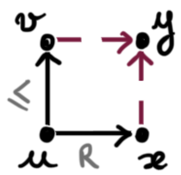
Relational Rules:

We can encode the ppties of being a preorder of $\leqslant$ in the rules (reflexive & trancitive)

$$\dfrac{\mathcal{R}, x \leqslant x \mid \Gamma \Rightarrow \Delta}{\mathcal{R} \mid \Gamma \Rightarrow \Delta}\ \text{refl} \qquad \dfrac{\mathcal{R}, x \leqslant y, y \leqslant z, x \leqslant z \mid \Gamma \Rightarrow \Delta}{\mathcal{R}, x \leqslant y, y \leqslant z \mid \Gamma \Rightarrow \Delta}\ \text{trans}$$

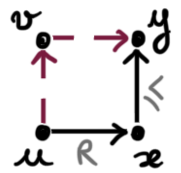
We also encode the confluence ppties relating $\leqslant$ and R

(FORW)



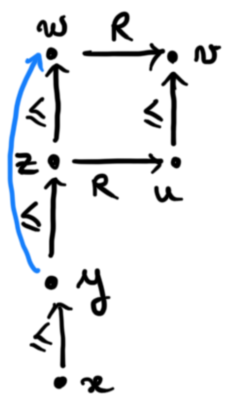
$$\dfrac{\mathcal{R}, u \leqslant v, uRx, x \leqslant y, vRy \mid \Gamma \Rightarrow \Delta}{\mathcal{R}, u \leqslant v, uRx \mid \Gamma \Rightarrow \Delta}\ \text{forw}^* \quad y \text{ fresh}$$

(BACK)



$$\dfrac{\mathcal{R}, uRx, x \leqslant y, u \leqslant v, vRy \mid \Gamma \Rightarrow \Delta}{\mathcal{R}, uRx, x \leqslant y \mid \Gamma \Rightarrow \Delta}\ \text{back}^* \quad v \text{ fresh}$$

Exercise Show $\mathsf{fIK} \vdash \Rightarrow x : (\Diamond A \to \Box B) \to \Box(A \to B)$



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{R}_4 \mid v:A \Rightarrow v:B, v:A}\ id}{\mathcal{R}_4 \mid v:A \Rightarrow v:B, w:\Diamond A}\ \Diamond_r \qquad \dfrac{\dfrac{}{\mathcal{R}_4 \mid v:A, v:B \Rightarrow v:B}\ id}{\mathcal{R}_4 \mid v:A, w:\Box B \Rightarrow v:B}\ \Box_l}{\mathcal{R}_4 \mid y:\Diamond A \to \Box B, v:A \Rightarrow v:B}\ \to_l}{\mathcal{R}_3 \mid y:\Diamond A \to \Box B, v:A \Rightarrow v:B}\ \text{trans}}{\mathcal{R}_2 \mid y:\Diamond A \to \Box B, v:A \Rightarrow v:B}\ \text{back}}{\mathcal{R}_1 \mid y:\Diamond A \to \Box B \Rightarrow u: A \to B}\ \to_r}{\mathcal{R}_0 \mid y:\Diamond A \to \Box B \Rightarrow y:\Box(A \to B)}\ \Box_r}{\Rightarrow x:(\Diamond A \to \Box B) \to \Box(A \to B)}\ \to_r$$