Intuitionistic Modal Logic

Sonia Marin

Lecture 5

$$
\begin{tikzcd}
 & \mathsf{IK} \vdash A \arrow[rr, leftrightarrow] \arrow[dl, "2"'] & & \vDash A \arrow[dd, shift right, "5"'] \\
\ell\mathsf{IK} + cut \vdash_{\Longrightarrow} x:A \arrow[dr, "3"'] & & & \\
 & \ell\mathsf{IK} \vdash_{\Longrightarrow} x:A \arrow[uu, "1"'] & & f\ell\mathsf{IK} \vdash_{\Longrightarrow} x:A \arrow[uu, shift right, "4"']
\end{tikzcd}
$$

Soundness & Completeness

$$
\begin{tikzcd}
f\ell\mathsf{IK} \vdash_{\Longrightarrow} x:A \arrow[r, shift left, "4"'] & \vDash A \arrow[l, shift left, "5"']
\end{tikzcd}
$$

Remark: We do not know of a way to interpret the fully labelled sequents as modal formulas.

Soundness

Let $G = \mathcal{R} \mid \Gamma \Rightarrow \Delta$ a fully labelled sequent.
$\mathcal{M} = (W, R^{\mathcal{M}}, \leqslant^{\mathcal{M}}, \Vdash)$ be a birelational model.

labels occurring in G

An interpretation of G in $\mathcal{M}$ is a mapping $[\![\cdot]\!] : \boxed{\ell(G)} \longrightarrow W$
s.t. whenever $xRy \in \mathcal{R}$, then $[\![x]\!] R^{\mathcal{M}} [\![y]\!]$
and whenever $x \leqslant y \in \mathcal{R}$, then $[\![x]\!] \leqslant^{\mathcal{M}} [\![y]\!]$

$\mathcal{M} \models G$: for any interpretation of G into $\mathcal{M}$ s.t.
if for all $x : A \in \Gamma$, we have $[\![x]\!] \Vdash A$
then there exists $y : B \in \Delta$ s.t. $[\![z]\!] \Vdash B$

Theorem $f\ell IK \vdash \Rightarrow x : A \xrightarrow[4]{\qquad} \models A$

i.e. if a fully labelled sequent G is provable then it is satisfied in every birelational model.

Proof By induction on the proof of G.

Want to show for each rule in $f\ell IK$ that whenever its premisses are valid then so is its conclusion.

$$\frac{\overbrace{\mathcal{R}, x \leqslant y, yRz \mid \Gamma \Rightarrow \Delta, z : A}^{G'}}{\underbrace{\mathcal{R} \mid \Gamma \Rightarrow \Delta, x : \Box A}_{G}}\ \Box_r$$

By contrapositive: Assume there is a birelational model $\mathcal{M} = (W, R^{\mathcal{M}}, \leqslant^{\mathcal{M}}, \Vdash)$ and an interpretation $[\![\cdot]\!]$ of G into $\mathcal{M}$ such that
for all $u : B \in \Gamma$, $[\![u]\!] \Vdash B$ and for all $z : C \in \Delta$, $[\![z]\!] \nVdash C$
and $[\![x]\!] \nVdash \Box A$

this means in particular
there is $v, w \in W$ s.t. $[\![x]\!] \leqslant^{\mathcal{M}} v\, R^{\mathcal{M}} w$ and $w \nVdash A$.

uses that y, z are fresh in G'

$\boxed{\text{Extend}}$ $[\![\cdot]\!]$ into $[\![\cdot]\!]'$ with $[\![y]\!] = v$ and $[\![z]\!] = w$
to get an interpretation of G' into $\mathcal{M}$
such that $\mathcal{M} \not\models G'$.

Completeness

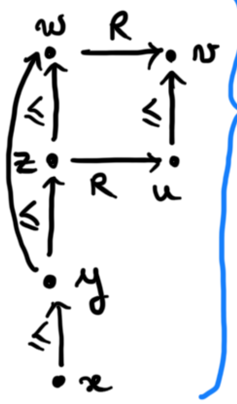
Let $\mathcal{G} = \mathcal{R} | \Gamma \Rightarrow \Delta$ a fully labelled sequent.

Pre-model $\mathfrak{M}_{\mathcal{G}}$ is given as $(\ell(\mathcal{G}), R^{\mathcal{G}}, \leq^{\mathcal{G}}, \Vdash)$ where

- $\leq^{\mathcal{G}} = \{(x,y) \mid x \leq y \in \mathcal{R}\}$
- $R^{\mathcal{G}} = \{(x,y) \mid xRy \in \mathcal{R}\}$
- $w \Vdash p$ iff $w:p \in \Gamma$.

Under adequate conditions (e.g. when rules refl, trans, back & forw have been applied exhaustively), $\mathfrak{M}_{\mathcal{G}}$ is an actual birelational model

Example



$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{R}_4 \mid v:A \Rightarrow v:B, v:A}\,id}{\mathcal{R}_4 \mid v:A \Rightarrow v:B, w:\Diamond A}\,\Diamond_r \quad \dfrac{\dfrac{}{\mathcal{R}_4 \mid v:A, v:B \Rightarrow v:B}\,id}{\mathcal{R}_4 \mid v:A, w:\Box B \Rightarrow v:B}\,\Box_l}{\mathcal{R}_4 \mid y: \Diamond A \to \Box B, v:A \Rightarrow v:B}\,\to_l}{\mathcal{R}_3 \mid y: \Diamond A \to \Box B, v:A \Rightarrow v:B}\,trans}{\mathcal{R}_2 \mid y: \Diamond A \to \Box B, v:A \Rightarrow v:B}\,back}{\mathcal{R}_1 \mid y: \Diamond A \to \Box B \Rightarrow u: A \to B}\,\to_r}{\mathcal{R}_0 \mid y: \Diamond A \to \Box B \Rightarrow y: \Box(A \to B)}\,\Box_r}{\Rightarrow x: (\Diamond A \to \Box B) \to \Box(A \to B)}\,\to_r
$$

Theorem $\;\mathsf{flIK} \vdash \; \Rightarrow x:A \quad \xleftarrow{⑤} \quad \models A$

Proof: By contrapositive suppose $\mathsf{flIK} \nvdash \underbrace{\mathcal{R} | \Gamma \Rightarrow \Delta}_{\mathcal{G}}$

- if no rules in flIK apply then $\mathfrak{M}_{\mathcal{G}} \not\models \mathcal{G}$.
- if a rule is s.t $\dfrac{\mathcal{G}_1 \;\; \cdots \;\; \mathcal{G}_n}{\mathcal{G}}$ (with $n \in \{0,1,2\}$) then by local soundness, $\not\models \mathcal{G}_i$ for some $1 \leq i \leq n$.

The process terminates using

(1) modal rules: reduce complexity bottom-up

$$
\dfrac{\mathcal{R}, x \leq y, yRz \mid \Gamma \Rightarrow \Delta, z:A}{\mathcal{R} \mid \Gamma \Rightarrow \Delta, x: \Box A}\,\Box_r
$$

(2) intuitionistic rules: can use standard IPL looptechniques

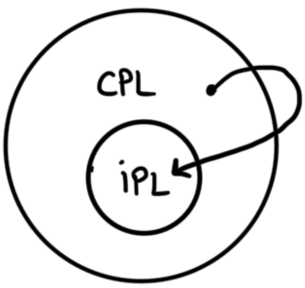
$$
\dfrac{\mathcal{R} \mid \Gamma, x: A \to B \Rightarrow \Delta, x:A \qquad \mathcal{R} \mid \Gamma, x:B \Rightarrow \Delta}{\mathcal{R} \mid \Gamma, x: A \to B \Rightarrow \Delta}\,\to_r
$$

$$
\begin{tikzcd}
 & \mathsf{IK} \vdash A \arrow[ld, "2"'] \arrow[rr, leftrightarrow] & & \vDash A \arrow[dd, shift right, "5"'] \\
\ell\mathsf{IK} + \mathrm{cut} \vdash_{\Rightarrow} x:A \arrow[rd, "3"'] & & & \\
 & \ell\mathsf{IK} \vdash_{\Rightarrow} x:A \arrow[uu, "1"'] & & f\ell\mathsf{IK} \vdash_{\Rightarrow} x:A \arrow[uu, shift right, "4"']
\end{tikzcd}
$$

We can now use the different corners of this diagram in whichever way is most convenient.

Translating classical into intuitionistic

from Lecture 1



Intuitionistic logic is "weaker" than classical logic as a fragment

But it is strong enough to emulate all classical reasoning

Negative translation: [Gödel-Gentzen]

$$
\begin{aligned}
p^N &:= \neg\neg p \\
(A \wedge B)^N &:= A^N \wedge B^N \\
(A \vee B)^N &:= \neg(\neg A^N \wedge \neg B^N) \\
(A \to B)^N &:= A^N \to B^N \\
\bot^N &:= \bot
\end{aligned}
$$

Theorem A is provable in CPL iff A^N is provable in IPL.

Proof: Can be proved via sequent system or via semantics.

Let us extend the translation to modalities:

$$(\Diamond A)^N := \neg \Box \neg A^N$$
$$(\Box A)^N := \Box A^N$$

Theorem A is provable in K iff A^N is provable in IK

Which makes sense as a corollary of the theorem for FOL.

Can also give a direct proof by induction on A, including

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{id}
\UnaryInfC{$xRy \mid x:\neg\neg\Box p, y:p, y:\neg p \Rightarrow y:p$}
\AxiomC{}
\RightLabel{$\bot l$}
\UnaryInfC{$xRy \mid x:\neg\neg\Box p, y:p, y:\bot \Rightarrow x:\bot$}
\RightLabel{$\to l$}
\BinaryInfC{$xRy \mid x:\neg\neg\Box p, y:p, y:\neg p \Rightarrow x:\bot$}
\RightLabel{$\Box l$}
\UnaryInfC{$xRy \mid x:\neg\neg\Box\neg p, y:p, x:\Box\neg p \Rightarrow x:\bot$}
\RightLabel{$\to r$}
\UnaryInfC{$xRy \mid x:\neg\neg\Box\neg p, y:p \Rightarrow x:\neg\Box\neg p$}
\AxiomC{}
\RightLabel{$\bot l$}
\UnaryInfC{$xRy \mid x:\bot, y:p \Rightarrow y:\bot$}
\RightLabel{$\to l$}
\BinaryInfC{$xRy \mid x:\neg\neg\Box\neg p, y:p \Rightarrow y:\bot$}
\RightLabel{$\to r$}
\UnaryInfC{$xRy \mid x:\neg\neg\Box\neg p \Rightarrow y:\neg p$}
\RightLabel{$\Box r$}
\UnaryInfC{$x:\neg\neg\Box\neg p \Rightarrow x:\Box\neg p$}
\RightLabel{$\to r$}
\UnaryInfC{$x: \neg\neg\Box\neg p \to \Box\neg p$}
\end{prooftree}
$$

back in 2022

This observation was the starting point of a conversation around the role/strength of the $\Diamond$ operator in IML.

The Proof Theory Blog

$\Gamma, A \vdash B \Rightarrow \Gamma \vdash A \to B$

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Brouwer meets Kripke: constructivising modal logic

Posted on August 19, 2022 by Anupam Das and Sonia Marin

Intuitionistic logic and modal logic each admit meaningful projections of classical theories. While intuitionistic logic can interpret classical logic by a host of ¬¬ translations, thus yielding computational interpretations, modal logic can be employed to understand notions of provability and truth. However, the enrichment of intuitionistic logic by modalities is far less canonical than its classical counterpart, giving rise to two prevailing approaches. In this post we survey the state of the art and revisit the role of intuitionistic modal logic in providing computational interpretations for classical modal logic.

$\boxed{\text{Logic IK – Recap}}$

$$\boxed{IPL \quad + \quad k_\Box \quad \Box(A\to B)\to(\Box A\to\Box B)} \quad \text{logic iK}$$

$$k_\Diamond : \quad \Box(A\to B)\to(\Diamond A\to\Diamond B)$$

$$n_\Diamond : \quad \Diamond\bot\to\bot$$

$$c_\Diamond : \quad \Diamond(A\vee B)\to(\Diamond A\vee\Diamond B)$$

$$i_{\Diamond\Box} : \quad (\Diamond A\to\Box B)\to\Box(A\to B)$$

$$\frac{A\in\Gamma}{\Gamma\vdash^{IK} A} \qquad \frac{A \text{ axiom}}{\Gamma\vdash^{IK} A} \qquad \frac{\Gamma\vdash^{IK} A \qquad \Gamma\vdash^{IK} A\to B}{\Gamma\vdash^{IK} B}\,\text{mp} \qquad \frac{\cdot\vdash^{IK} A}{\Gamma\vdash^{IK}\Box A}\,\text{nec}$$

IK does not prove more <u>propositional</u> theorems than IPL

Similarly, it seemed to have been accepted in the literature IK did not prove more <u>$\Box$-only / $\Diamond$-free</u> theorems than iK

<u>In fact it does</u> — and Simpson / Grefe knew about it

↳ in particular $iK \nvdash \neg\neg\Box\neg p\to\Box\neg p$

The Proof Theory Blog

$\Gamma, A\vdash B \Rightarrow \Gamma\vdash A\to B$

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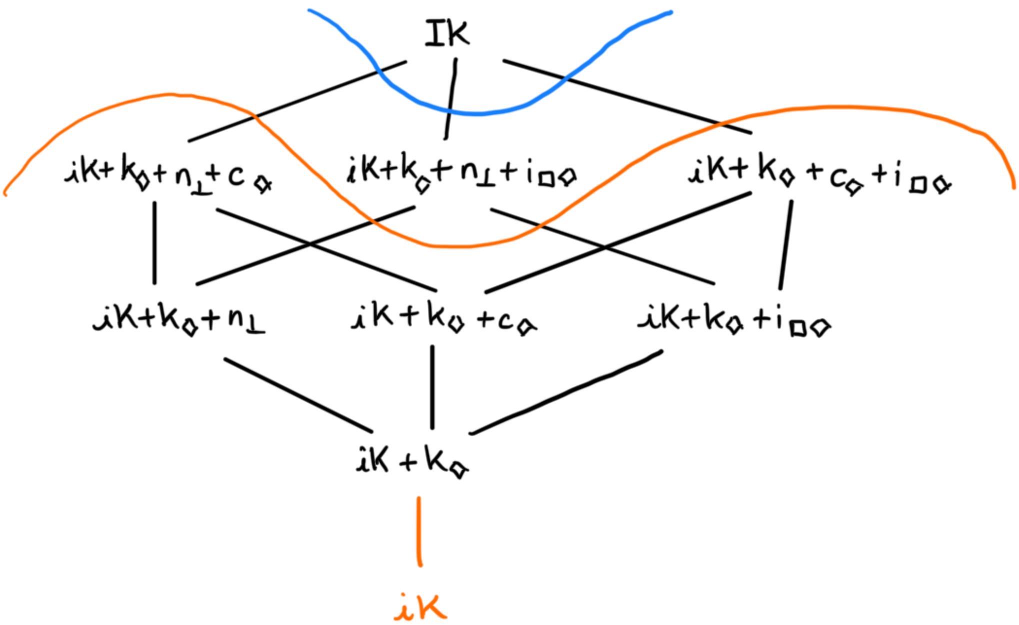
Diamond-free parts of intuitionistic modal logics

Posted on October 3, 2025 by Anupam Das, Ian Shillito and Jim de Groot

A post by Anupam and Sonia in 2022 ignited a surge of interest from the modal logic and proof theory communities: until then, it was commonly believed that different inuitionistic versions of modal logic K agreed on their $\Diamond$-free parts, but this turns out to be false. After 23 comments, 3 papers, 1 further blog post, and 1 hunt for an elusive PhD thesis, we can now finally provide a classification of $\Diamond$-free parts.

The IML community became interested in understanding the <u>fragments of logic IK</u> better.

From their <u>axiomatic definition</u> on the one hand.



e.g. what is the correct semantics for these sub-logics?
[de Groot, Shillito & Clouston, 2025]

From their <u>semantic definition</u> on the other hand

e.g. how to axiomatize the logic obtained by removing <u>backward confluence</u>?
[Balbiani, Gao, Gencer & Olivetti, 2024]