

# LSS 2025: Computability and Incompleteness

V. Provability Predicates

Gödel's Second Incompleteness Theorem

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# Outline

- 1 Introduction
- 2 Provability Predicates
- 3 Gödel's Second Incompleteness Theorem
- 4 (Non-)Implications



# Last Time...

## Representability

- ▶ All recursive functions are representable in extensions of  $\mathcal{Q}$

## Arithmetic Cannot be Captured

- ▶ The diagonalisation function is computable
- ▶ So any candidate “theorem-hood” notion can be turned against itself
  - ▶ “I am true iff I am not a theorem”
- ▶ Truth is not definable in arithmetic (Tarski)
- ▶ Arithmetic is not axiomatisable (Gödel)



# Peano Arithmetic

Called, variously: *PA* (*Johnstone*), *Z* (*B&J*), *S* (*Mendelson*).

Take  $\mathcal{Q}$ , and add induction:

- ▶ If  $P$  is a formula with  $x$  free, then the universal closure of

$$P(0) \wedge (\forall m. P(m) \Rightarrow P(s(m))) \Rightarrow (\forall n. P(m))$$

is an axiom.

(Where  $P(a)$  means  $P$  with  $x$  replaced by  $a$ .)

The result is a formal system with an infinite number of axioms.

- ▶ However, the axioms are still decidable.



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# Proofs are Computably Checkable

A proof in a formal system is a sequence of formulas such that every formula in the sequence is

- ▶ an instance of an axiom; or
- ▶ is the result of applying a rule of inference to one or more formulas earlier in the sequence

For human consumption, we usually indicate a non-axiom's forebears explicitly.

But we could just check all possible earlier formulas.



# Proofs are Arithmetisable

Already know how to map

- ▶ formulas into numbers
- ▶ lists of numbers into numbers.

Can therefore turn a proof into a number.

Checking this number is really a proof is computable, hence representable in extensions of  $\mathcal{Q}$ .

Given formula  $A$ , can also check that the last formula in a proof is equal to  $A$ .

Thus

$$\textit{Proof}(p, \ulcorner A \urcorner) = p \text{ is a proof of } A$$

is definable.



# A Provability Predicate

Let  $Provable(n) \stackrel{\text{def}}{=} (\exists p. Proof(p, n))$

Write  $\Box A$  for  $Provable(\ulcorner A \urcorner)$ .

## Important Properties of Provability:

- ▶ if  $\vdash A$  then  $\vdash \Box A$
- ▶  $\vdash \Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$
- ▶  $\vdash \Box A \Rightarrow \Box(\Box A)$

In  $\mathcal{Z}$  the above can all be proved; as can

- ▶ if  $\vdash_{\mathcal{Z}} \Box A$  then  $\vdash_{\mathcal{Z}} A$





# Provability Does Not Define Theorem-Hood

Last time, we proved the indefinability of theorem-hood.

Definability required

$$\vdash_{\mathcal{T}} Thm(nt(n)) \quad \text{iff} \quad \vdash_{\mathcal{T}} gn^{-1}(n) \quad (1)$$

$$\vdash_{\mathcal{T}} \neg Thm(nt(n)) \quad \text{iff} \quad \not\vdash_{\mathcal{T}} gn^{-1}(n) \quad (2)$$

Provability ( $\square$ ) only gives us (1).

So what happens if we replay the proof of indefinability with  $\square$ ?



# The Gödel Sentence

We have a  $G$  such that

$$\vdash_Z G \iff \neg \Box G \quad (1)$$

- This is the **Gödel sentence** for our theory.

We also know that

$$\vdash_Z G \text{ iff } \vdash_Z \Box G \quad (2)$$

If  $Z$  is consistent, then:

$G$  is not a theorem of  $Z$ .

- If it were, then  $\vdash_Z G$ . So,  $\vdash_Z \Box G$  by (2). But also,  $\vdash_Z \neg \Box G$  by (1), making  $Z$  inconsistent.

$\neg G$  is not a theorem of  $Z$ .

- If it were, then  $\vdash_Z \Box G$  by (1). Then  $\vdash_Z G$  by (2). Again making  $Z$  inconsistent.



# Gödel's First Incompleteness Theorem Concretely

As long as our logic  $\mathcal{T}$  is strong enough to give us

$$\vdash_{\mathcal{T}} G \quad \text{iff} \quad \vdash_{\mathcal{T}} \Box G$$

we know

*If  $\mathcal{T}$  is consistent, then  $\nvdash_{\mathcal{T}} G$  and  $\nvdash_{\mathcal{T}} \neg G$*

In other words,  $G$  demonstrates  $\mathcal{T}$ 's incompleteness.

Moreover, we do know that  $\vdash_{\mathcal{T}} G \iff \neg \Box G$

- ▶ This says that  $G$  is true iff  $G$  is not provable.
- ▶ Having just proved  $G$ 's unprovability, we can conclude  $G$  is true.



# Henkin's Formula

On one hand,  $G$  says that  $G$  isn't derivable.

Diagonalisation also gives us  $H$  such that

$$\vdash_{\mathcal{T}} H \iff \Box H$$

or

*$H$  says that  $H$  is derivable*

But is  $H$  true?



# Löb's Theorem

By far the weirdest result of the course:

*If  $\vdash_{\mathcal{T}} \Box A \Rightarrow A$ , then  $\vdash_{\mathcal{T}} A$*

Can also write:

$$\Box(\Box A \Rightarrow A) \Rightarrow \Box A$$

which is the the axiom for modal provability logic.

(Why does provability “correspond” to a binary relation that is transitive and well-founded?)



# Proof of Löb's Theorem

Theorem: if  $\vdash_{\mathcal{T}} \Box A \Rightarrow A$ , then  $\vdash_{\mathcal{T}} A$

Diagonalise formula  $\Box x \Rightarrow A$ , giving  $L$  such that

- |    |  |   |
|----|--|---|
| 1  | $\vdash_{\mathcal{T}} L \iff (\Box L \Rightarrow A)$                       |   |
| 2  | $\vdash_{\mathcal{T}} L \Rightarrow (\Box L \Rightarrow A)$                | (bicond elimination)                    |
| 3  | $\vdash_{\mathcal{T}} \Box(L \Rightarrow (\Box L \Rightarrow A))$          | (PP1)                                   |
| 4  | $\vdash_{\mathcal{T}} \Box L \Rightarrow \Box(\Box L \Rightarrow A)$       | (PP2)                                   |
| 5  | $\vdash_{\mathcal{T}} \Box L \Rightarrow (\Box \Box L \Rightarrow \Box A)$ | (PP2 on right)                          |
| 6  | $\vdash_{\mathcal{T}} \Box L \Rightarrow \Box A$                           | (PP3 eliminates $\Box \Box L$ )         |
| 7  | $\vdash_{\mathcal{T}} \Box L \Rightarrow A$                                | ( $\Box A \Rightarrow A$ by assumption) |
| 8  | $\vdash_{\mathcal{T}} L$   | (7,1)                                   |
| 9  | $\vdash_{\mathcal{T}} \Box L$  | (PP1)                                   |
| 10 | $\vdash_{\mathcal{T}} A$   | (7,9)                                   |



# Löb's Theorem Proves the Henkin Sentence

Henkin sentence is  $\vdash_{\mathcal{T}} H \iff \Box H$

If that's provable, so too is  $\vdash_{\mathcal{T}} \Box H \Rightarrow H$ .

By Löb's Theorem:  $\vdash_{\mathcal{T}} H$

So the sentence that “says of itself that it is provable”, is indeed true.



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# Provability Gives us Arithmetisation of Consistency

Write  $\perp$  for  $0 \neq 0$ . (Recall that  $\vdash \perp \Rightarrow A$  for any  $A$ .)

Write  $\text{Con}_{\mathcal{T}}$  for  $\neg \Box \perp$  (“false” is not provable).

- ▶ Consistency was “actually” simultaneous derivation of  $A$  and  $\neg A$  for some  $A$
- ▶ But the two are equivalent.



# Consistency is Unprovable (Sketchy Version)

Want to show

$$\vdash_{\mathcal{T}} \text{Con}_{\mathcal{T}} \Rightarrow G$$

Then,  $\text{Con}_{\mathcal{T}}$  can't be derivable, because if it were,  $G$  would be too.

We know that  $G$  “means” ‘ $G$  is not derivable’.

Gödel's First Incompleteness Theorem says

*If  $\mathcal{T}$  is consistent, then  $G$  is not derivable.*

But that's just what we want to prove!

- ▶ Just have to be able to carry out proof of Gödel's First Incompleteness Theorem in  $\mathcal{T}$

Done



# Consistency is Unprovable (Löb Version)

Suppose we did have  $\vdash_{\mathcal{T}} \text{Con}_{\mathcal{T}}$ , or  $\vdash_{\mathcal{T}} \neg \Box \perp$ .

Then get:  $\vdash_{\mathcal{T}} \Box \perp \Rightarrow \perp$

- ▶ by propositional principle of proving anything from a false assumption

Löb's Theorem then says  $\vdash_{\mathcal{T}} \perp$  (false is derivable after all!)

A contradiction, so consistency is not provable.

Done



# Consistency is Unprovable (non-Löb PP Version)

Recall that  $G$  demonstrates  $\mathcal{T}$ 's incompleteness (is unprovable).

Now want to argue that if  $\mathcal{T}$  extends  $\mathcal{Z}$ , then

$$\vdash_{\mathcal{T}} \text{Con}_{\mathcal{T}} \Rightarrow G$$

(if  $\text{Con}_{\mathcal{T}}$  were provable,  $G$  would be too).

- ▶ Have (provability property):  $\vdash_{\mathcal{T}} \Box G \Rightarrow \Box \Box G$
- ▶ Thus (diagonal property of  $G$ ):  $\vdash_{\mathcal{T}} \Box G \Rightarrow \Box \neg G$ 
  - ▶ “if I can prove  $G$ , then I can also prove  $\neg G$ ”
- ▶ So,  $\vdash_{\mathcal{T}} \Box G \Rightarrow \Box \perp$
- ▶ Diagonal property of  $G$ :  $\vdash_{\mathcal{T}} \neg G \Rightarrow \Box \perp$
- ▶ Contrapositively:  $\vdash_{\mathcal{T}} \neg \Box \perp \Rightarrow G$

Done



# Gödel's Second Incompleteness Theorem

If  $\mathcal{T}$  is at least as powerful as  $\mathcal{Z}$ , then it cannot simultaneously:

- ▶ Be consistent
- ▶ Prove its own consistency



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# Would a Consistency Proof of $\mathcal{T}$ in $\mathcal{T}$ Be Convincing?

Imagine we are doubtful about  $\mathcal{T}$ .

A consistency proof would be reassuring.

But if that proof is carried out in  $\mathcal{T}$  too,  
how does that assuage our doubts?

- If it could be done in a small part of  $\mathcal{T}$ , maybe...



# Consistency is Possible by Other Means

Peano Arithmetic was proved consistent by Gentzen.  
( $\mathcal{Q}$ 's consistency follows too.)

He didn't do it in PA, but used a different logical system.

Nor was his system stronger than PA; just different.





# Yikes, An Infinite Regress Awaits!

If we can't prove our interesting systems consistent except by recourse to other systems, this is a neverending process!



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If we can't prove our interesting systems consistent except by recourse to other systems, this is a neverending process!

## So what?

- ▶ We have the same problem whenever we set up our logical systems; we have to start with some set of axioms.
- ▶ “We don't need Gödel to tell us that we cannot accept a proof in one formal system only on the basis of proof in another formal system.”—Franzén



# Note

Consistent systems don't have to prove true theorems.



# My Own Self-Doubt-Casting Sentence

If anyone says

“ $X$  because of Gödel’s Theorem”

or

“Thanks to Gödel’s Theorem,  $X$ ”

or variants of the same...

...they’re talking nonsense.



# My Own Self-Doubt-Casting Sentence

If anyone says

“ $X$  because of Gödel’s Theorem”

or

“Thanks to Gödel’s Theorem,  $X$ ”

or variants of the same...

...they’re talking nonsense.

(To a first approximation.)



## Examples from Franzén

- ▶ *Religious people claim that all answers are to be found in the Bible or in whatever text they use. That means the Bible is a complete system, so Gödel seems to indicate it cannot be true. And the same may be said of any religion which claims, as they all do, a final set of answers.*
- ▶ *As Gödel demonstrated, all consistent formal systems are incomplete, and all complete formal systems are inconsistent. The U.S. Constitution is a formal system, after a fashion. The Founders made the choice of incompleteness over inconsistency, and the Judicial Branch exists to close that gap of incompleteness.*
- ▶ *Gödel demonstrated that any axiomatic system must be either incomplete or inconsistent, and inasmuch as Ayn Rand's philosophy of Objectivism claims to be a system of axioms and propositions, one of these two conditions must apply.*
- ▶ *Nonstandard models and Gödel's incompleteness theorem point the way to God's freedom to change both the structure of knowing and the objects known.*

# Mathematics Floundering in a Relativistic Sea?

We can extend  $\mathcal{T}$  by adding either  $G$  or  $\neg G$  as a new axiom.

The resulting theory will be consistent if  $\mathcal{T}$  was.

- How do we pick which one to take?

For  $\mathcal{Z}$  (PA), we know that  $G \iff \text{Con}_{\mathcal{Z}}$ .

- We also know  $\text{Con}_{\mathcal{Z}}$  (Gentzen), so we should pick  $\mathcal{Z} + G$ .

For more complicated systems (e.g., ZFC set theory),  
“ordinary mathematics” does not necessarily know their consistency.

- but systems  $\text{ZFC} + \neg \text{Con}_{\text{ZFC}}$  are uninteresting



# Gödel and AI

Lucas:

*However complicated a machine we construct, it will, if it is a machine, correspond to a formal system, which in turn will be liable to the Gödel procedure for finding a formula unprovable in that system. This formula the machine will be unable to produce as true, although a mind can see that it is true.*

False.

- ▶ The Gödel formula is equivalent to the consistency of the system; it is not true in general.
- ▶ The “human mind” is not known to have any special ability to determine the consistency of arbitrary formal systems.

Also, see Franzén for more on Penrose’s various arguments.





# Summary

## Provability Predicates

- ▶ Logical theories as strong as  $\mathcal{Z}$  can capture the notion of provability.
- ▶ Modal axioms must characterise the putative modality ( $\Box$ )
- ▶ Löb: if  $\vdash_{\mathcal{T}} \Box A \Rightarrow A$ , then  $\vdash_{\mathcal{T}} A$

## Gödel's Second Incompleteness Theorem

- ▶ A system as strong as  $\mathcal{Z}$  cannot both be consistent and prove its own consistency.

## Be Careful Out There



# Course Summary

## Computability

- ▶ Turing Machines and Recursive Functions are equivalent.
  - ▶ No extant computational model is more powerful
- ▶ Uncomputable problems exist (Halting Problem, notably)

## Logic and Incompleteness

- ▶ Validity in FOL is undecidable (by reduction to Halting Problem)
- ▶ Logics with minimal arithmetic can **represent** computable functions.
- ▶ By **diagonalisation** of formulas (a computable procedure):
  - ▶ arithmetic truth is undecidable;
  - ▶ no theory can be all three of consistent, complete, axiomatisable
- ▶ No theory extending  $\mathcal{Z}$  can prove its own consistency

