C02 Computational Complexity

Time and Space Complexity
Algorithm vs Problem Complexity
Big O Notation
Examples

Computational Complexity

Key computational resources:

- **Time**
- **Space**
- Energy, communications, I/O, samples...

Computational complexity is the study of how problem size affects resource consumption (how it *scales*). Distinguish:

- **Algorithm Complexity**: for a given algorithm / implementation
- **Problem Complexity**: for *any* algorithm that solves the problem
  - Inherit difficulty of the problem (Computational Complexity Theory)

Algorithm Complexity

- Identify ***n***, the number that characterizes the problem size.
  - Number of pixels on screen
  - Number of elements to be sorted
  - etc.
- Study the algorithm to determine how resource consumption changes as a function of ***n***.
- The *content* of the input, not just its size, can be important. Can study:
  - **Worst** case (the worst input of size ***n***)
  - **Best** case (the best input of size ***n***)
  - **Average** case (average of distribution of inputs of size ***n***)
  - **Amortized** analysis (amortized cost over a sequence of ***n*** typical operations)
    - Useful for an operation with state that occasionally has an expensive step

Big O Notation

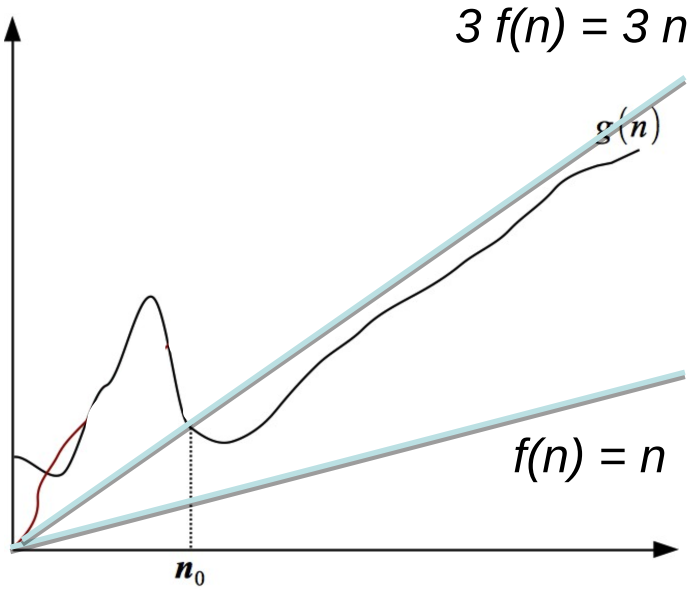
Suppose we have a problem of size *n* that takes *g(n)* time to execute in the average case.

We say:

$$g(n) \in O(f(n))$$

iff there exists constants $c > 0$ and $n_0 > 0$ such that for all $n > n_0$:

$$g(n) \leq c \times f(n)$$



Time complexity

In analysis of algorithm time complexity, we are interested in the number of “**elementary operations/statements**” (not μs).

- Simple statements are constant time.
- Remember the factor *c* in *O(f(n))*.
- Beware: Library/subroutine calls can have arbitrary complexity.

Example: Greatest Up To

Find the greatest element ≤ x in an unsorted sequence of *n* elements (or else return `null`).

Two approaches:

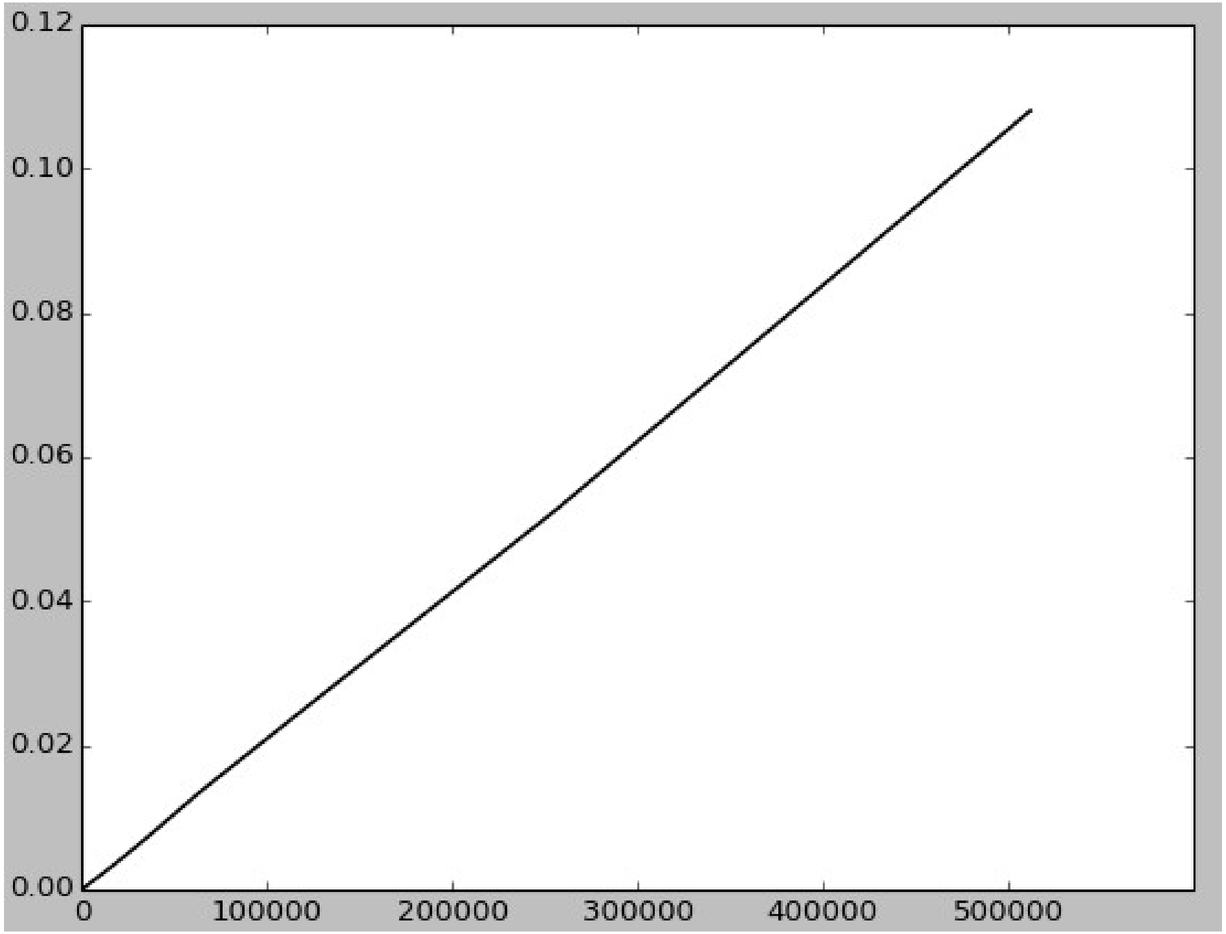
- a) search the unsorted sequence; or
- b) first sort the sequence, then search the sorted sequence.

Unsorted Greatest Up To

```
static Integer unsortedFind(int x, List<Integer> uList) {
    Integer best = null;
    for (var e : uList) {
        if (e == x)
            return e;
        if (e <= x && (best == null || e > best))
            best = e;
    }
    return best;
}
```

Analysis

- If we're lucky, `uList[0] == x`.
- Worst case?
  - `uList = {x – n, ..., x – 2, x – 1}`
  - $f(n) = 6n$, so $O(n)$

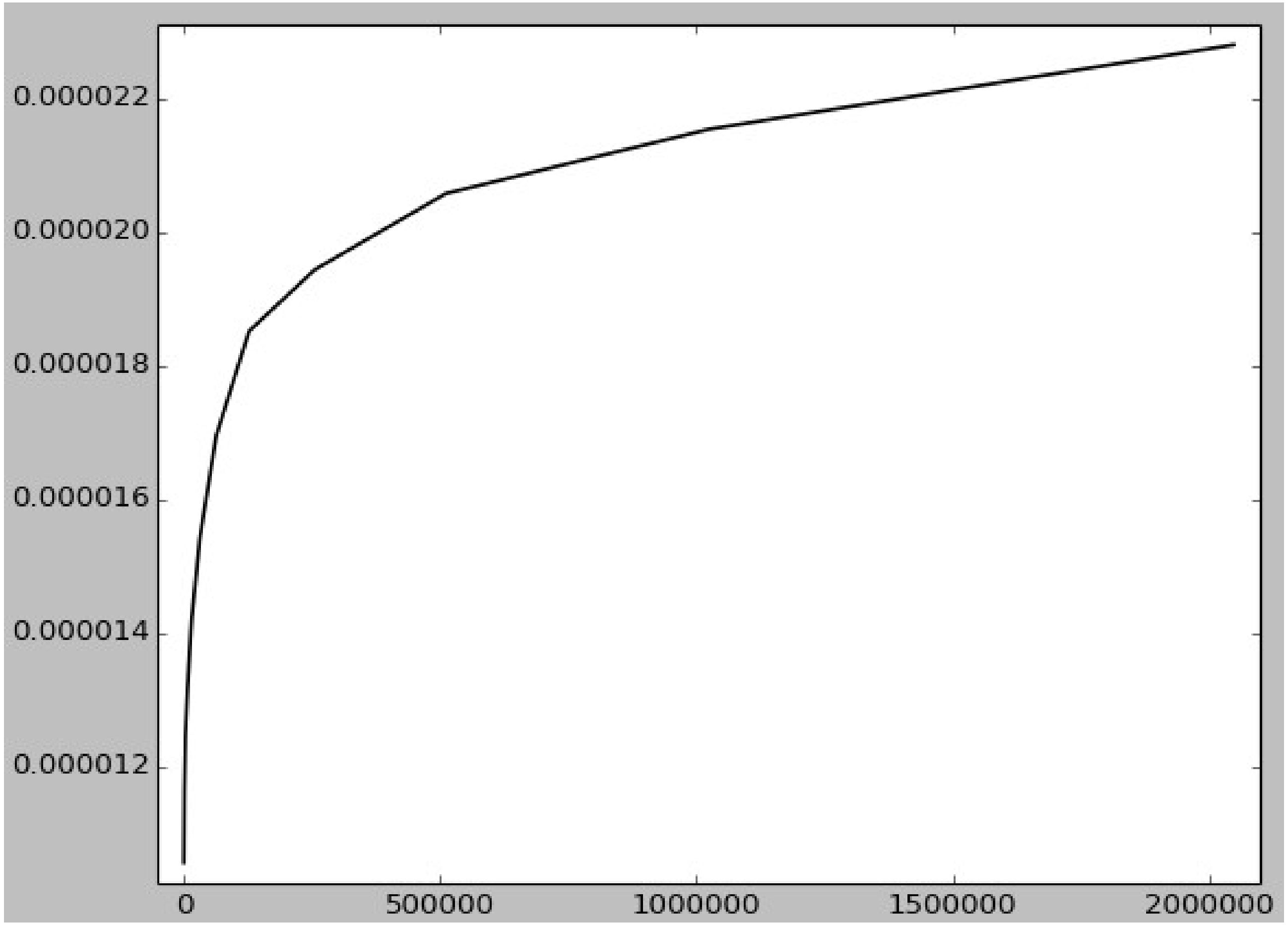

0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
100000
200000
300000
400000
500000

Sorted Greatest Up To

```
static Integer sortedFind(int x, ArrayList<Integer> sList) {
    if (sList.isEmpty() || sList.get(0) > x)
        return null;
    int lower = 0;
    int upper = sList.size();  // one past the end
    while (upper - lower > 1) {
        int mid = (lower + upper) / 2;
        if (sList.get(mid) <= x)
            lower = mid;
        else
            upper = mid;
    }
    return sList.get(lower);
}
```

Analysis

- How many iterations of the loop?
- Initially, `upper – lower = n`.
- The difference is halved in every iteration.
- Can halve it at most $log_2(n)$ times before it becomes 1.
- $f(n) = a\ log_2(n) + b$, so $O(log(n))$.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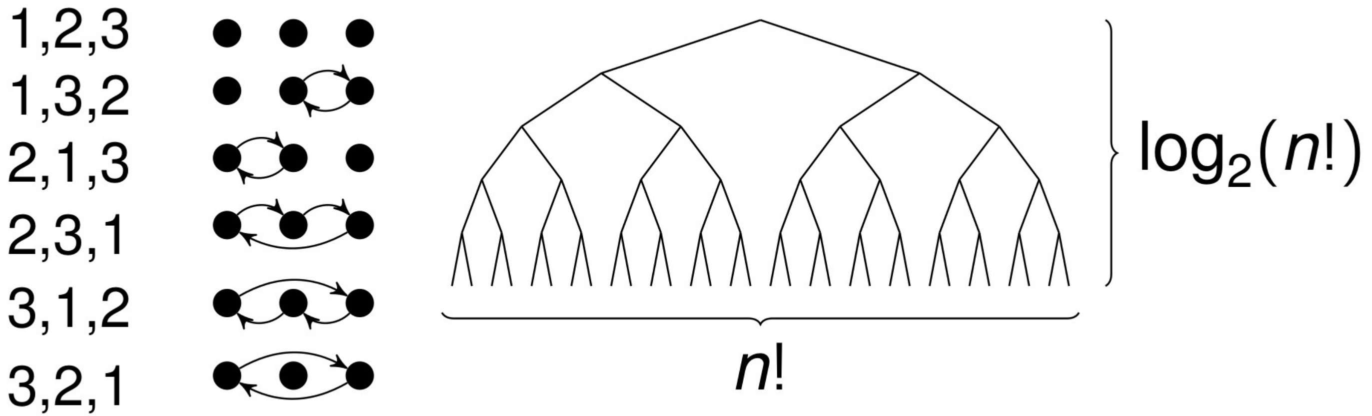
0.000018
0.000016
0.000014
0.000012
0
500000
1000000
1500000
2000000

Problem complexity

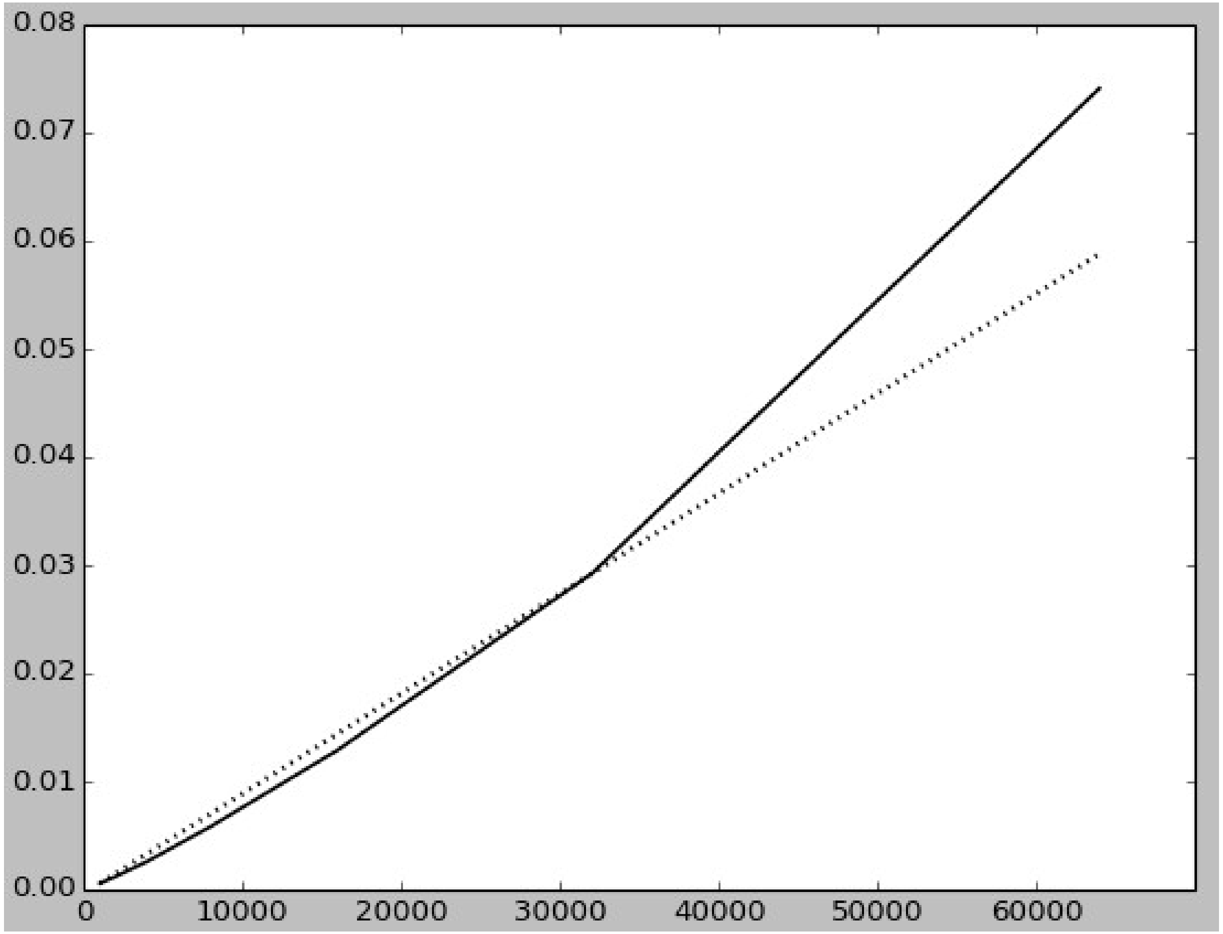
The complexity of a **problem** is the resources (time, memory, etc) that any algorithm *must* use, in the worst case, to solve the problem, as a function of instance size.

How fast can you sort?

Any sorting algorithm that uses only pair-wise comparisons needs *O(n log(n))* comparisons in the worst case.



log(n!) = *log(1)* + *log(2)* + … + *log(n)* ≤ *n log(n)* for large enough *n*.


0.08
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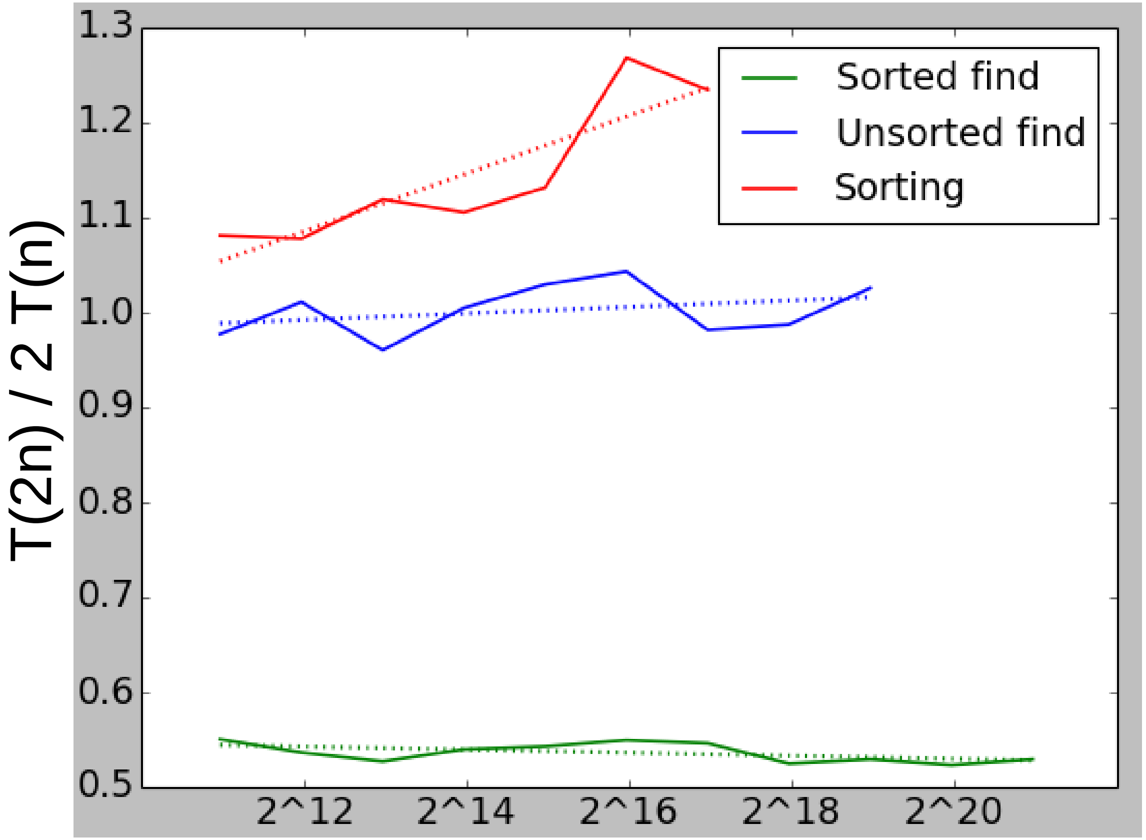
50000
60000

Rate of Growth



Example: Summing a List

Consider summing a list of size n…

```
public int sum(List<Integer> list) {
    int result = 0;
    for (var i : list) {
        result += i;
    }
    return result;
}
```

Linear time, $O(n)$

Example: Minimum Difference

Note: $n - 1 + n - 2 + \ldots 2 + 1 = n(n-1)/2$

```
public int minDiff(List<Integer> values) {
    int min = Integer.MAX_VALUE;                              1
    for (int i = 0; i < values.size(); i++) {                 n
        for (int j = i + 1; j < values.size(); j++) {         n(n – 1)/2
            int diff = values.get(i) - values.get(j);         n(n – 1)/2
            if (Math.abs(diff) < min)                         n(n – 1)/2
                min = Math.abs(diff);                         <= n(n – 1)/2
        }
    }
}
```

$$S(n) = 1 + n + 4\,(n(n-1)/2)$$
$$= 1 + n + 2\,n^2 - 2n$$
$$= 2n^2 - n + 1 \in O(n^2)$$

More Examples

- Constant $O(1)$
  - Time to perform an addition; swap two elements in an array; compare two numbers
  - Time to do any of the above 1000 times.
- Logarithmic $O(\log(n))$
  - Time to find an element in a B-Tree (self-balancing tree)
- Linear $O(n)$
  - Time to find an element in a list; sum a list of numbers
  - Find the min/max in a list?
- $O(n \log(n))$
  - Time to sort using mergesort
- Quadratic $O(n^2)$
  - Time to compare n elements with each other pair-wise.

Caution

“Premature optimization is the root of all evil in programming.”

(C.A.R. Hoare)

Scaling behaviour becomes important when problems become large, or when they need to be solved very frequently.