Fusing filters with Integer Linear Programming

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I don’t want to write *this*

```
      DO 10 I = 1, SIZE(XS)
          SUM1 = SUM1 + XS(I)
          IF (XS(I) .GT. 0) THEN
              SUM2 = SUM2 + XS(I)
          END IF
10    CONTINUE

      DO 20 I = 1, SIZE(XS)
          NOR1(I) = XS(I) / SUM1
          NOR2(I) = XS(I) / SUM2
20    CONTINUE
```

I'd rather write this

```
sum1 = fold   (+) 0     xs
nor1 = map    (/ sum1) xs
ys   = filter (>  0)   xs
sum2 = fold   (+) 0     ys
nor2 = map    (/ sum2) xs
```

But I also want speed

- Naive compilation: one loop for each combinator
- We need fusion!

Vertical fusion

```
sum1 = fold   (+) 0      xs  -- loop 1
nor1 = map    (/ sum1) xs  -- loop 2
ys   = filter (>  0)    xs  -- loop 3
sum2 = fold   (+) 0      ys  -- loop 4
nor2 = map    (/ sum2) xs  -- loop 5
```

Vertical fusion

```
sum1 = fold      (+) 0      xs  -- loop 1
nor1 = map       (/ sum1) xs  -- loop 2
sum2 = filterFold
       (> 0)  (+) 0      xs  -- loop 3
nor2 = map       (/ sum2) xs  -- loop 4
```

Horizontal fusion

```
sum1 = fold      (+) 0      xs   -- loop 1
nor1 = map       (/ sum1) xs   -- loop 2
sum2 = filterFold
         (> 0)  (+) 0      xs   -- loop 3
nor2 = map       (/ sum2) xs   -- loop 4
```

Horizontal fusion

```
sum1 = fold      (+) 0        xs   -- loop 1

(nor1, sum2)

    = mapFilterFold (/ sum1)

        (> 0)  (+) 0        xs   -- loop 2

nor2 = map       (/ sum2) xs   -- loop 3
```

Finished

```
sum1 = fold    (+) 0     xs  -- loop 1

(nor1, sum2)

     = mapFilterFold (/ sum1)

       (> 0)  (+) 0      xs  -- loop 2

nor2 = map     (/ sum2) xs  -- loop 3
```

Multiple choices

- What if we applied the fusion rules in a different order?
- There are far too many to try all of them, but…

Order matters

```
sum1 = fold   (+) 0     xs  -- loop 1
nor1 = map    (/ sum1) xs  -- loop 2
ys   = filter (>  0)    xs  -- loop 3
sum2 = fold   (+) 0     ys  -- loop 4
nor2 = map    (/ sum2) xs  -- loop 5
```

Order matters

```
sum1 = fold    (+) 0      xs  -- loop 1
nor1 = map     (/ sum1) xs  -- loop 2
sum2 = filterFold
       (>  0)  (+) 0      xs  -- loop 3
nor2 = map     (/ sum2) xs  -- loop 5
```

Order matters

```
(sum1, sum2)  = foldFilterFold
                (+) 0
      (>  0)    (+) 0     xs   -- loop 1
nor1 = map      (/ sum1) xs   -- loop 2
nor2 = map      (/ sum2) xs   -- loop 3
```

Order matters

```
(sum1, sum2)  = foldFilterFold
                (+) 0
      (>  0)    (+) 0     xs  -- loop 1
nor1 = map      (/ sum1) xs  -- loop 2
nor2 = map      (/ sum2) xs  -- loop 3
```

Order matters

```
(sum1, sum2)  = foldFilterFold
                (+) 0
      (>  0)  (+) 0     xs  -- loop 1
(nor1, nor2) = mapMap
     (/ sum1) (/ sum2) xs  -- loop 2
```

Order matters

```
(sum1, sum2)  = foldFilterFold
                (+) 0
      (>  0)  (+) 0     xs  -- loop 1
(nor1, nor2) = mapMap
     (/ sum1) (/ sum2) xs  -- loop 2
```

Which order?

- Finding the *best* order is the hard part.
- That's why we use…

Integer Linear Programming!

Minimise	$y - x$	*Objective*
Subject to	$0 \leq x \leq 2$	*Constraints*
	$0 \leq y \leq 2$	
	$x + 2y \geq 3$	
Where	$x : \mathbb{Z}$	*Variables*
	$y : \mathbb{Z}$	

Integer Linear Programming!

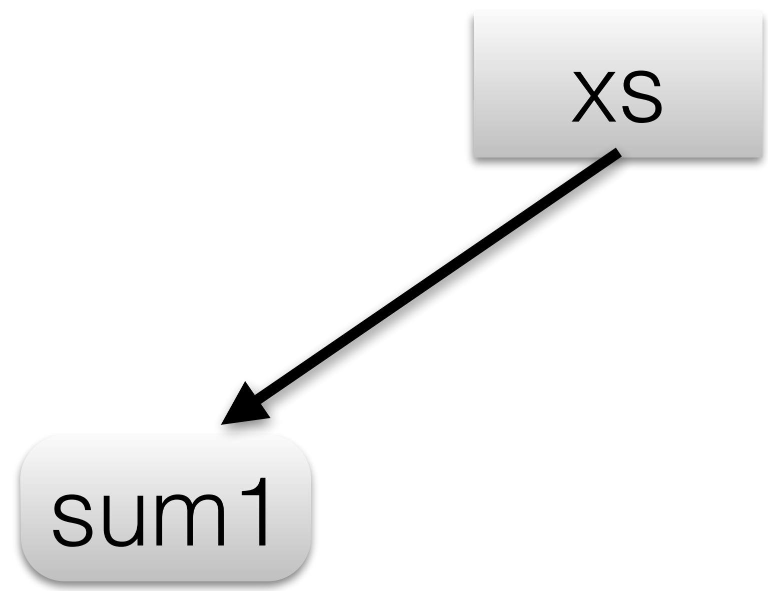
Minimise	$y - x$		*Objective*
Subject to	$0 \leq x \leq 2$		*Constraints*
	$0 \leq y \leq 2$		
	$x + 2y \geq 3$		
Where	$x : \mathbb{Z}$	$= 2$	*Variables*
	$y : \mathbb{Z}$	$= 1$	

Create a graph

xs

```
sum1 = fold   (+) 0     xs
nor1 = map    (/ sum1) xs
ys   = filter (>  0)   xs
sum2 = fold   (+) 0     ys
nor2 = map    (/ sum2) xs
```

```
sum1 = fold   (+) 0     xs
nor1 = map    (/ sum1) xs
ys   = filter (>  0)   xs
sum2 = fold   (+) 0     ys
nor2 = map    (/ sum2) xs
```



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sum2 = fold   (+) 0     ys
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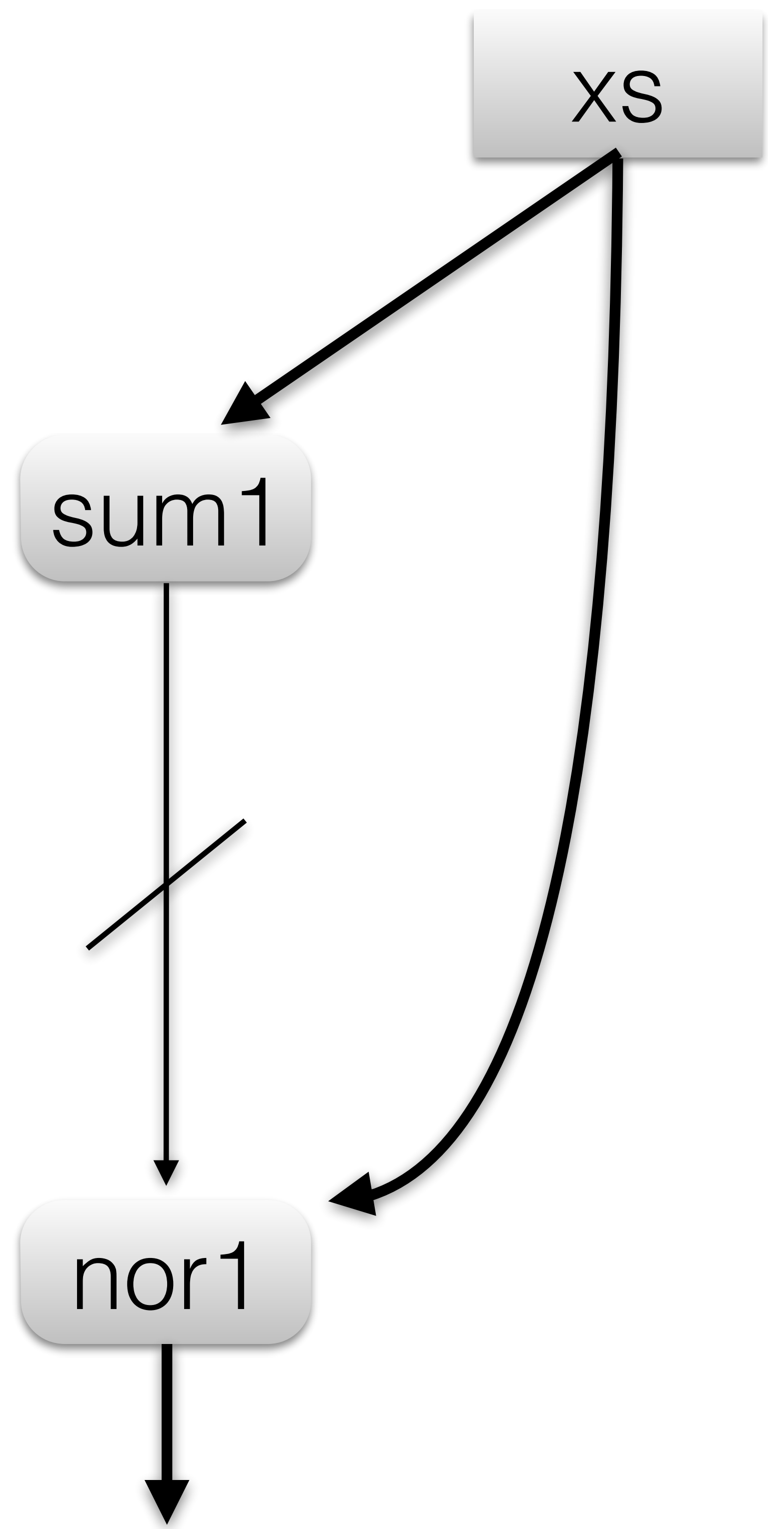

```
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nor2 = map    (/ sum2) xs
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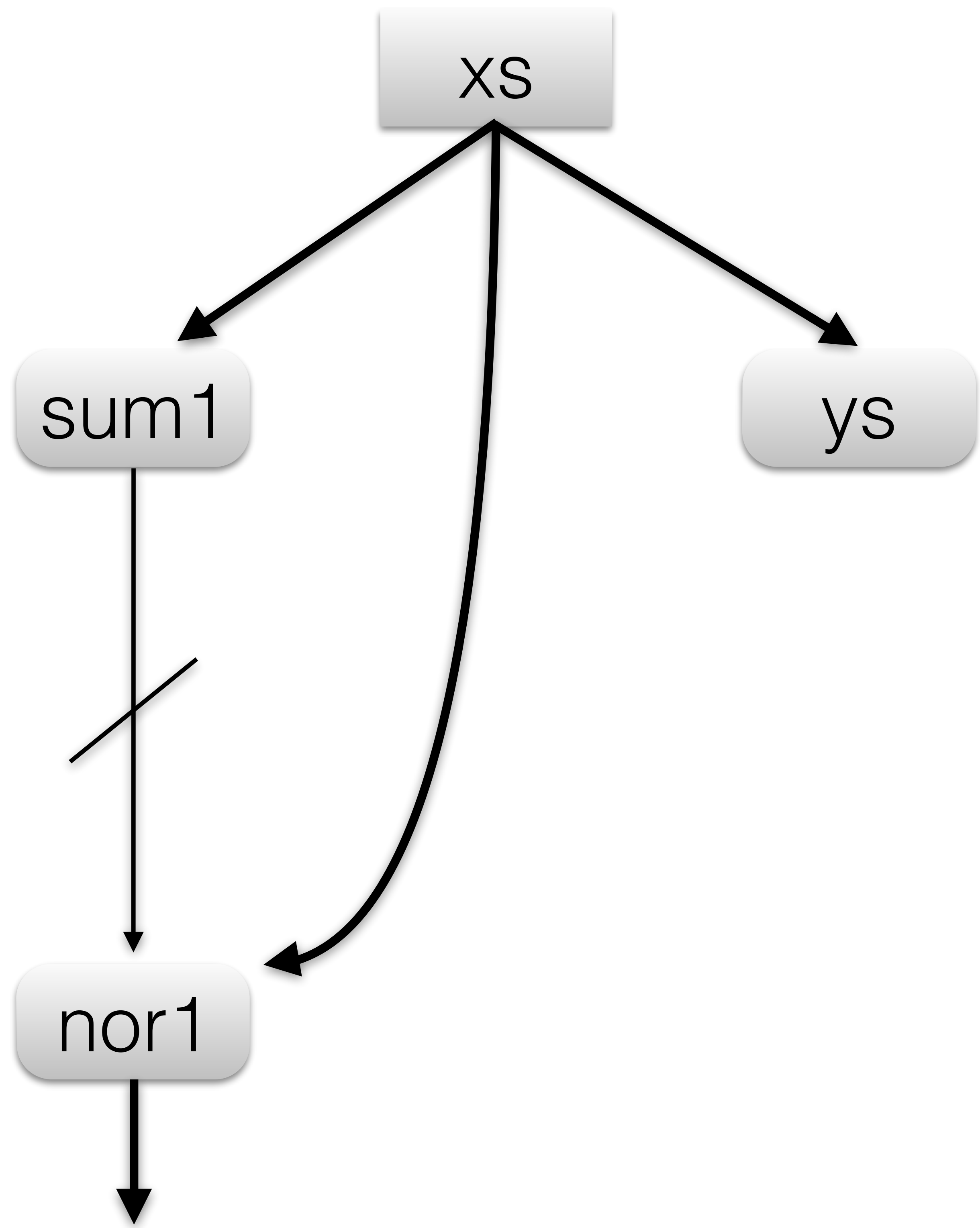

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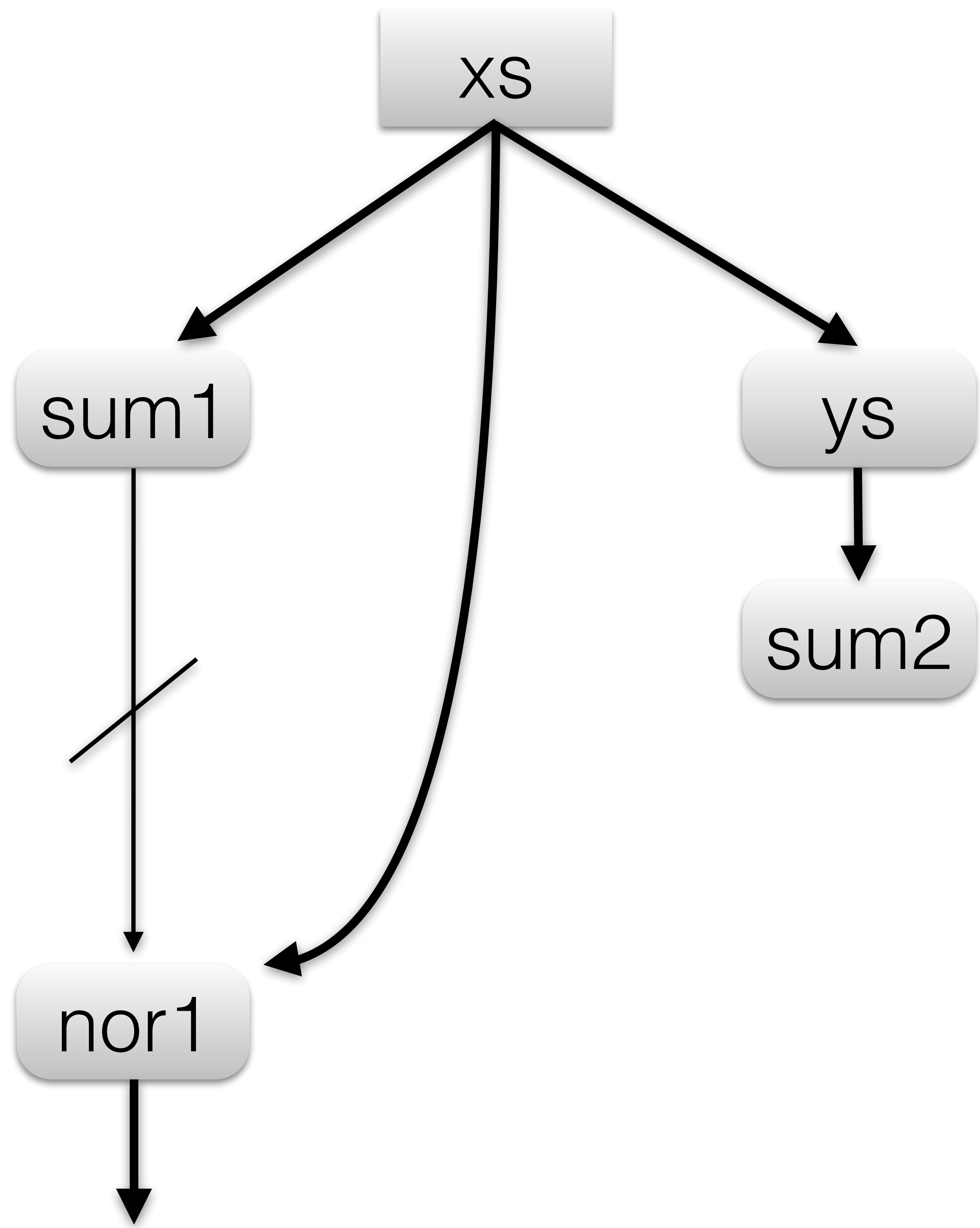

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ys   = filter (>  0)   xs
sum2 = fold   (+) 0     ys
nor2 = map    (/ sum2) xs
```
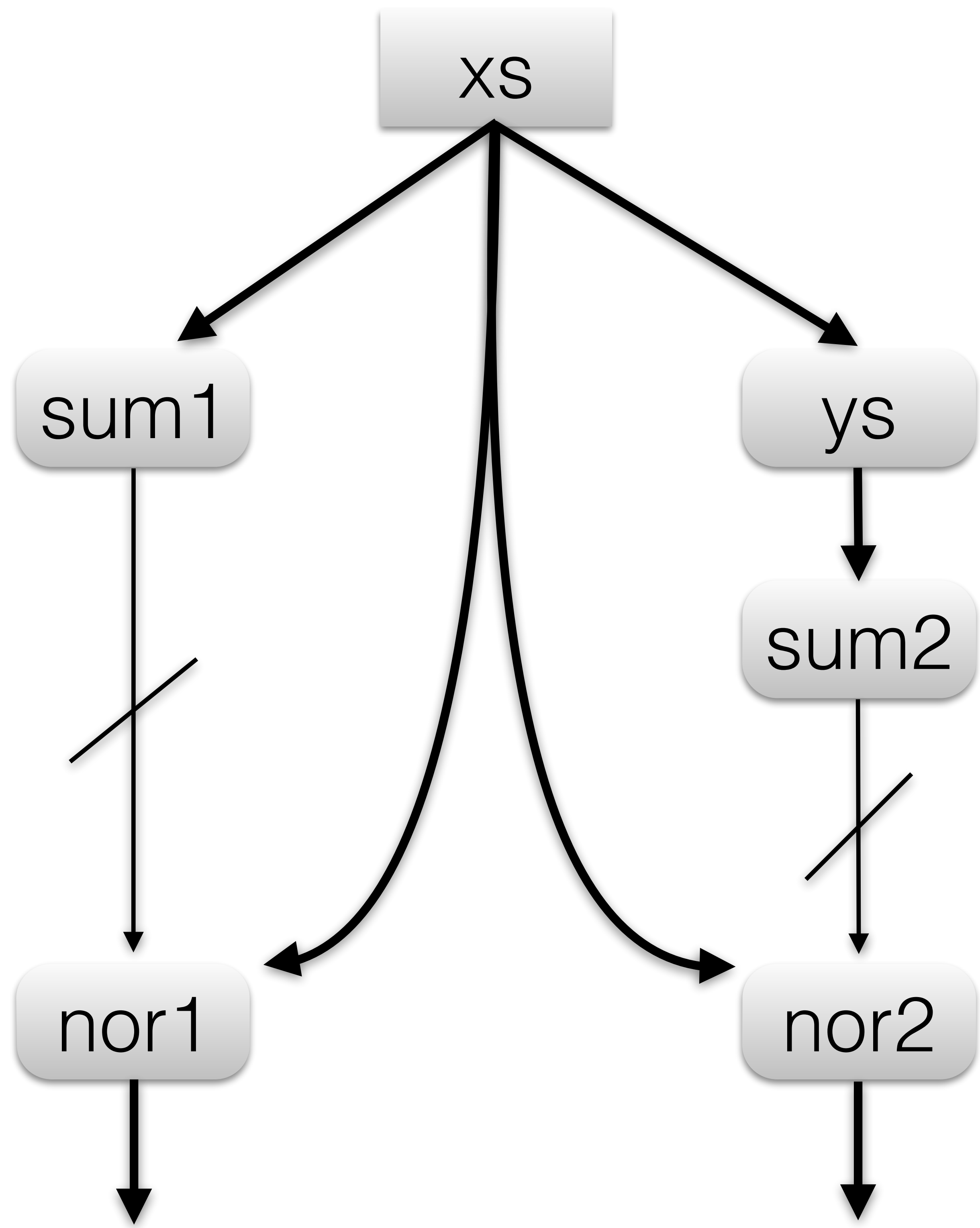
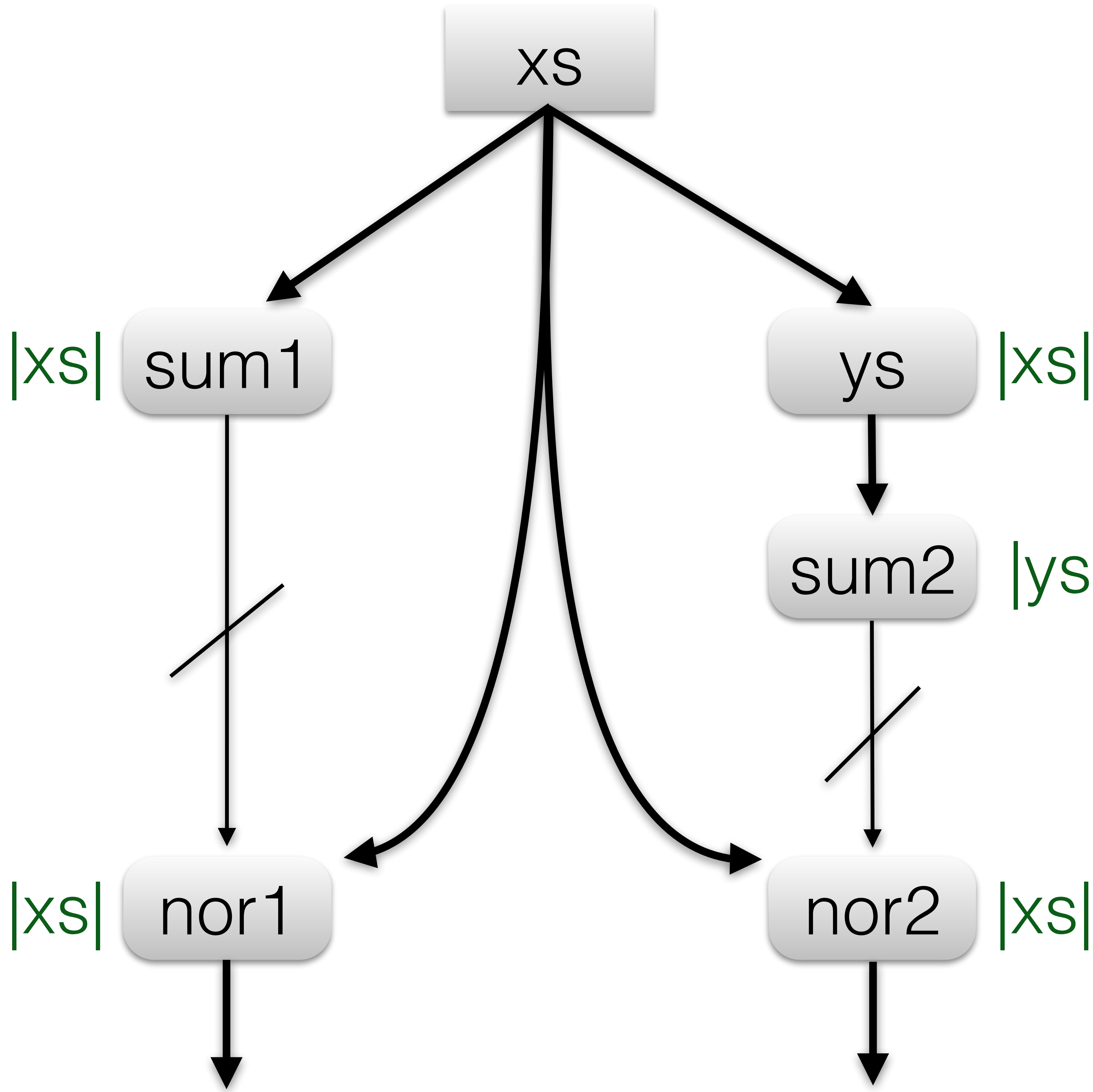
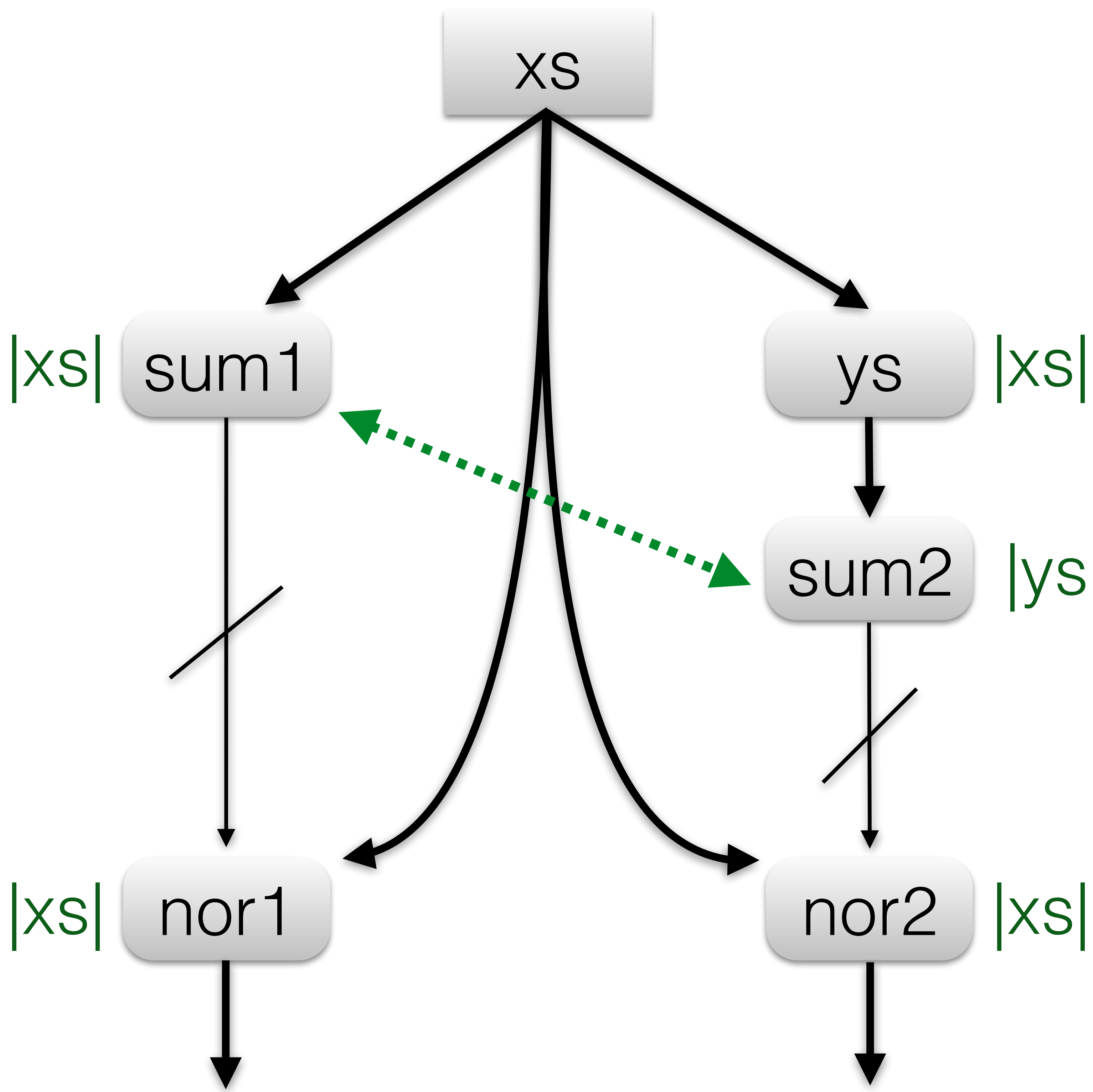



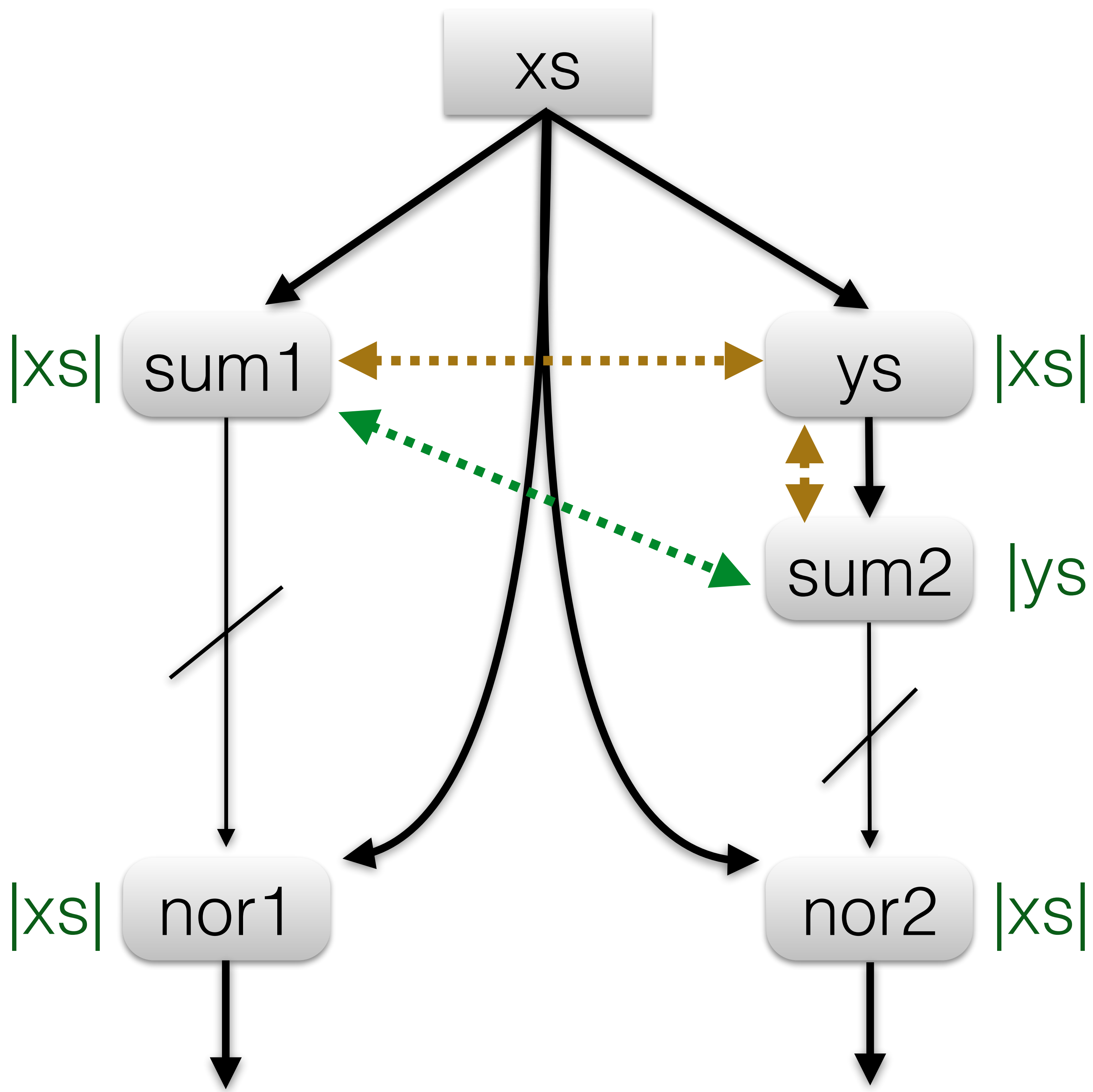
Different size loops



Different size loops



Different size loops



Filter constraint

Minimise …

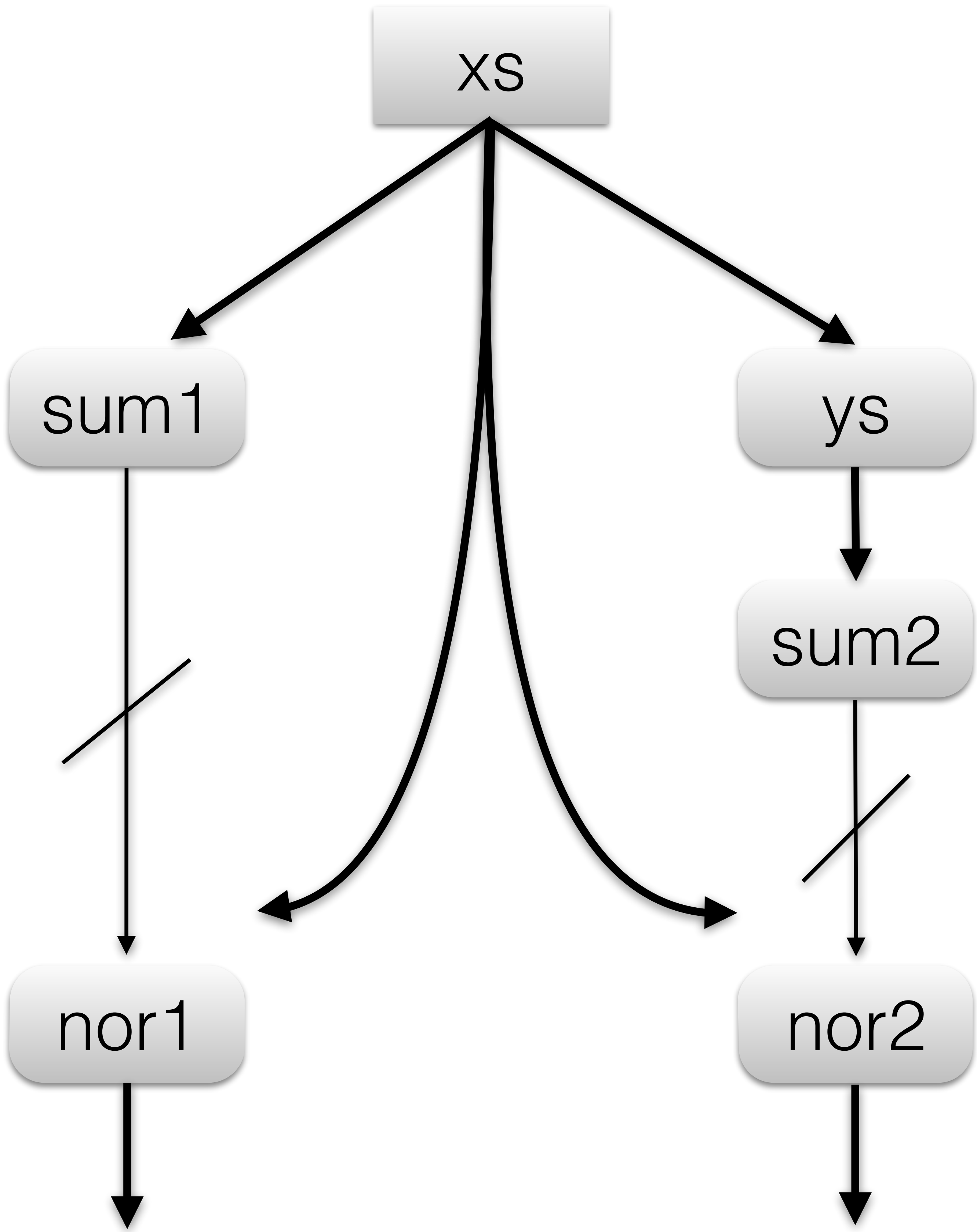
Subject to …

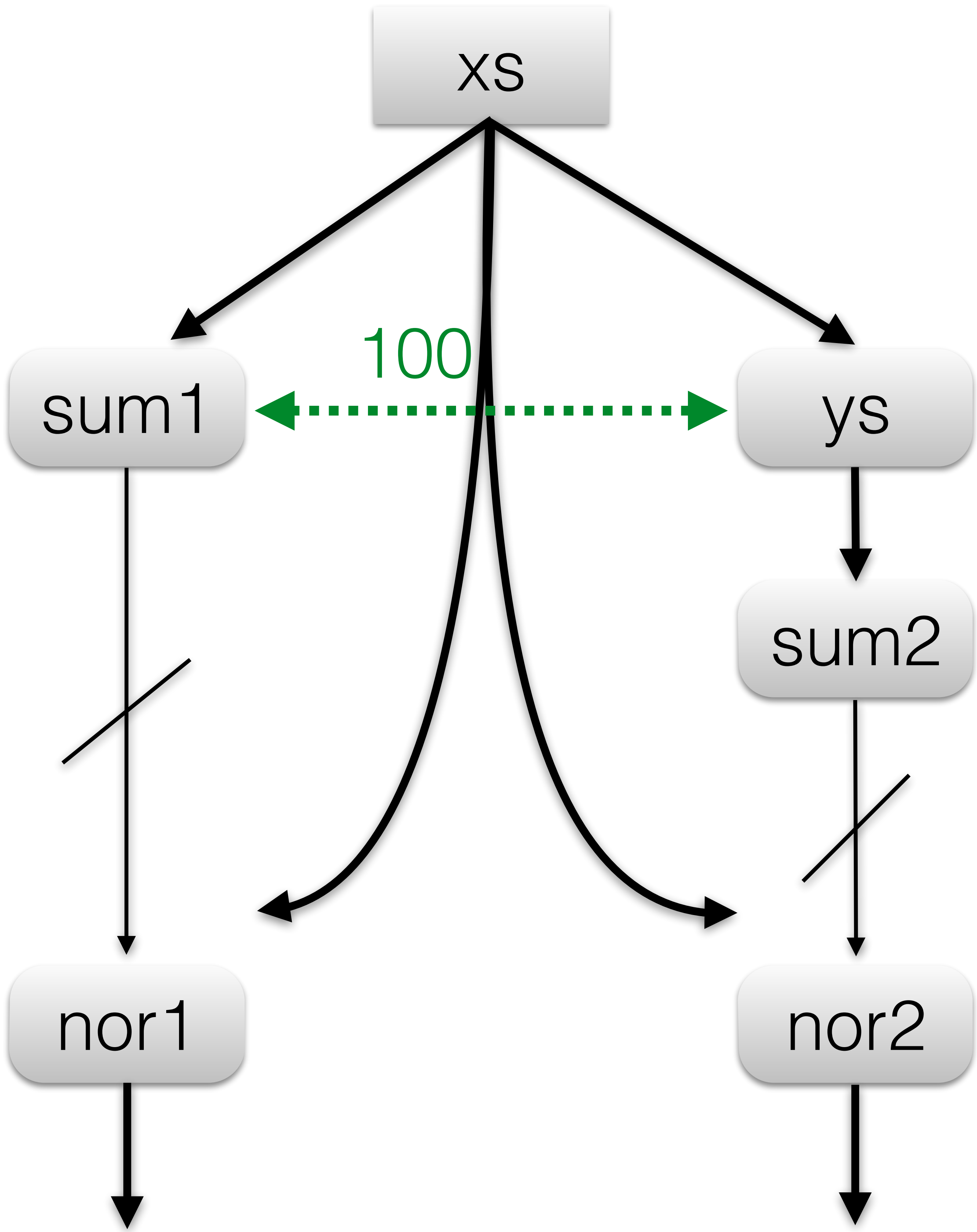
$$f(\texttt{sum1}, \texttt{ys}) \leq f(\texttt{sum1}, \texttt{sum2})$$

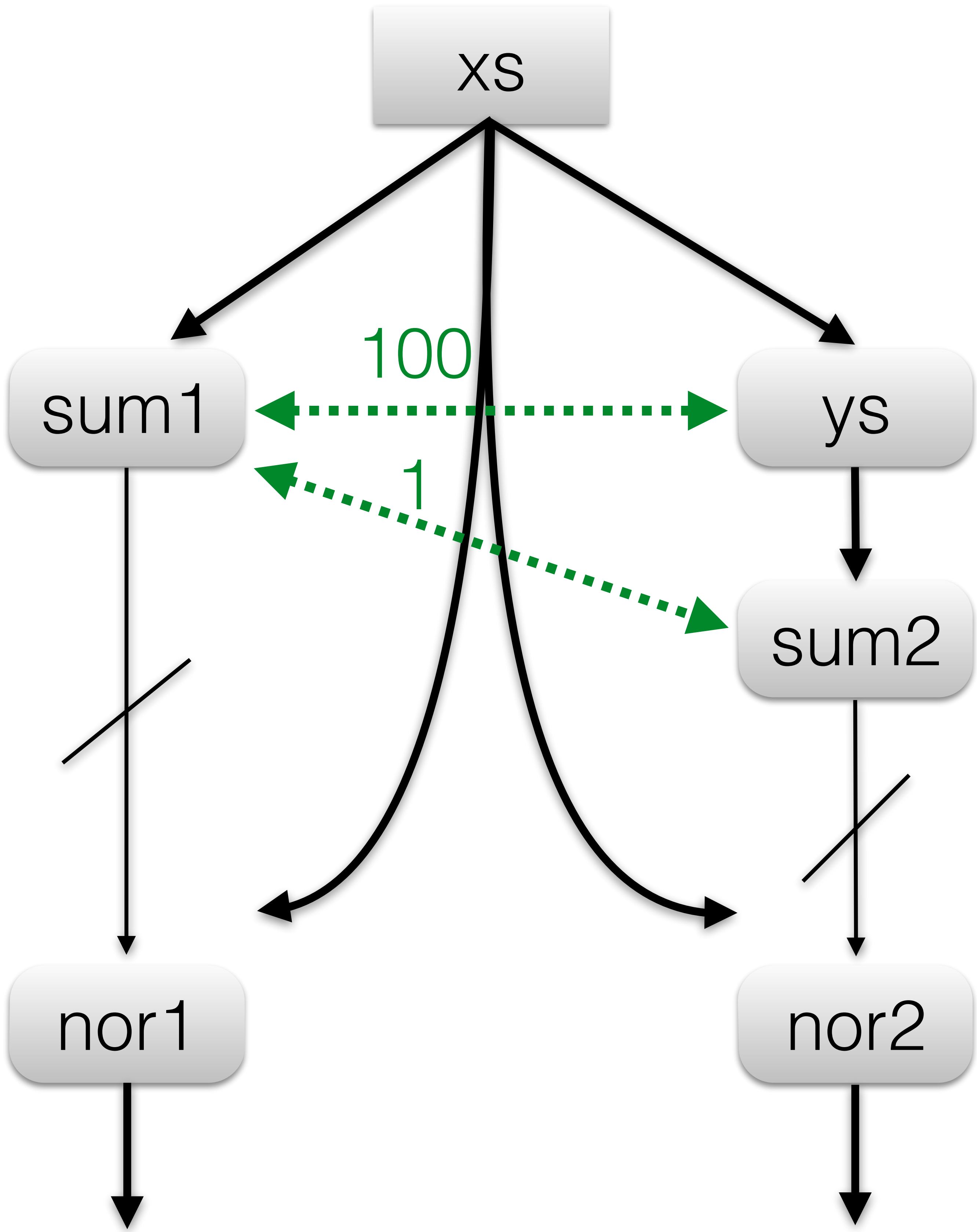
$$f(\texttt{sum2}, \texttt{ys}) \leq f(\texttt{sum1}, \texttt{sum2})$$

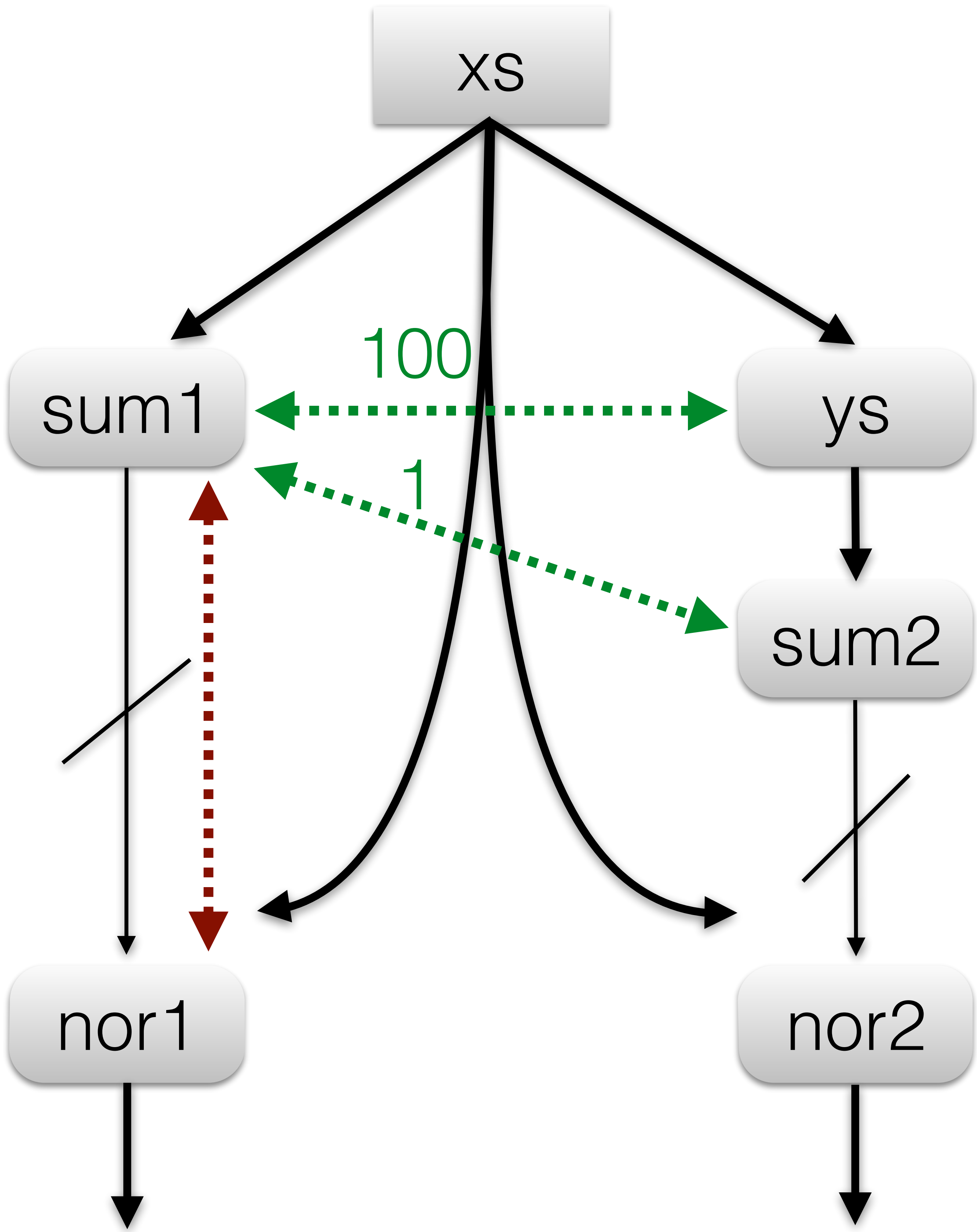
$f(\texttt{a},\texttt{b}) = 0$ iff `a` and `b` are fused together

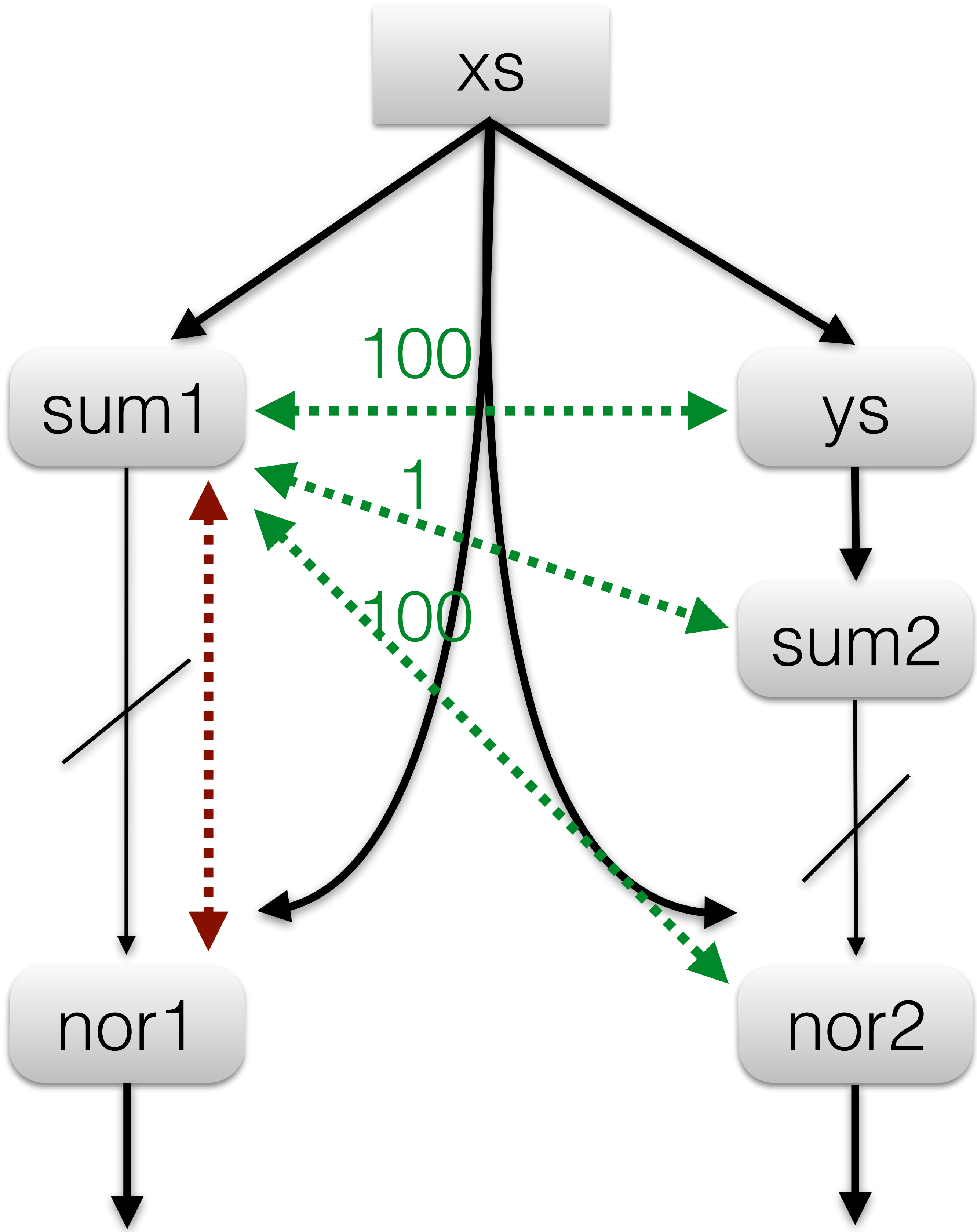
Objective function


xs
sum1
ys
sum2
nor1
nor2

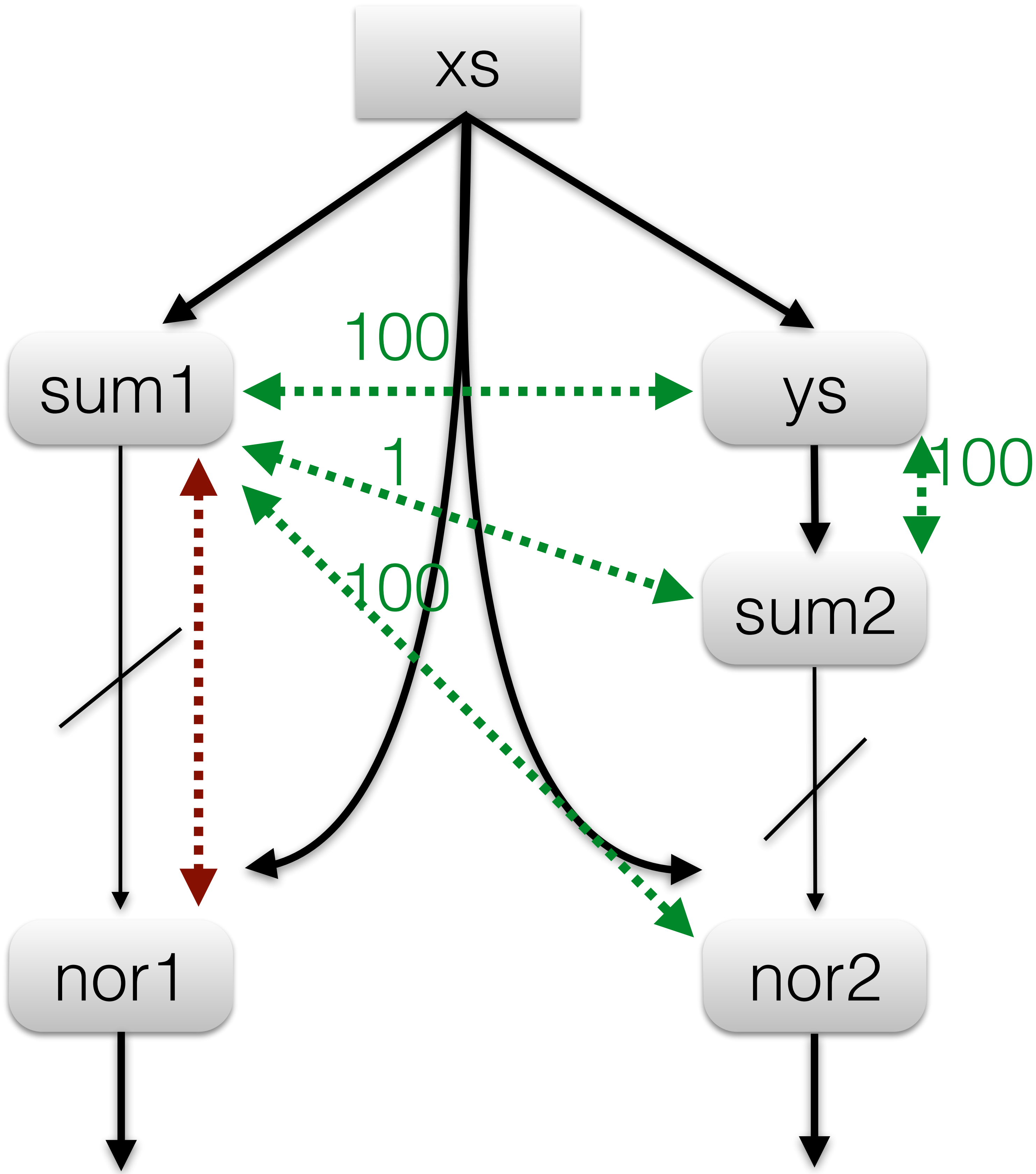

xs
100
sum1
ys
sum2
nor1
nor2

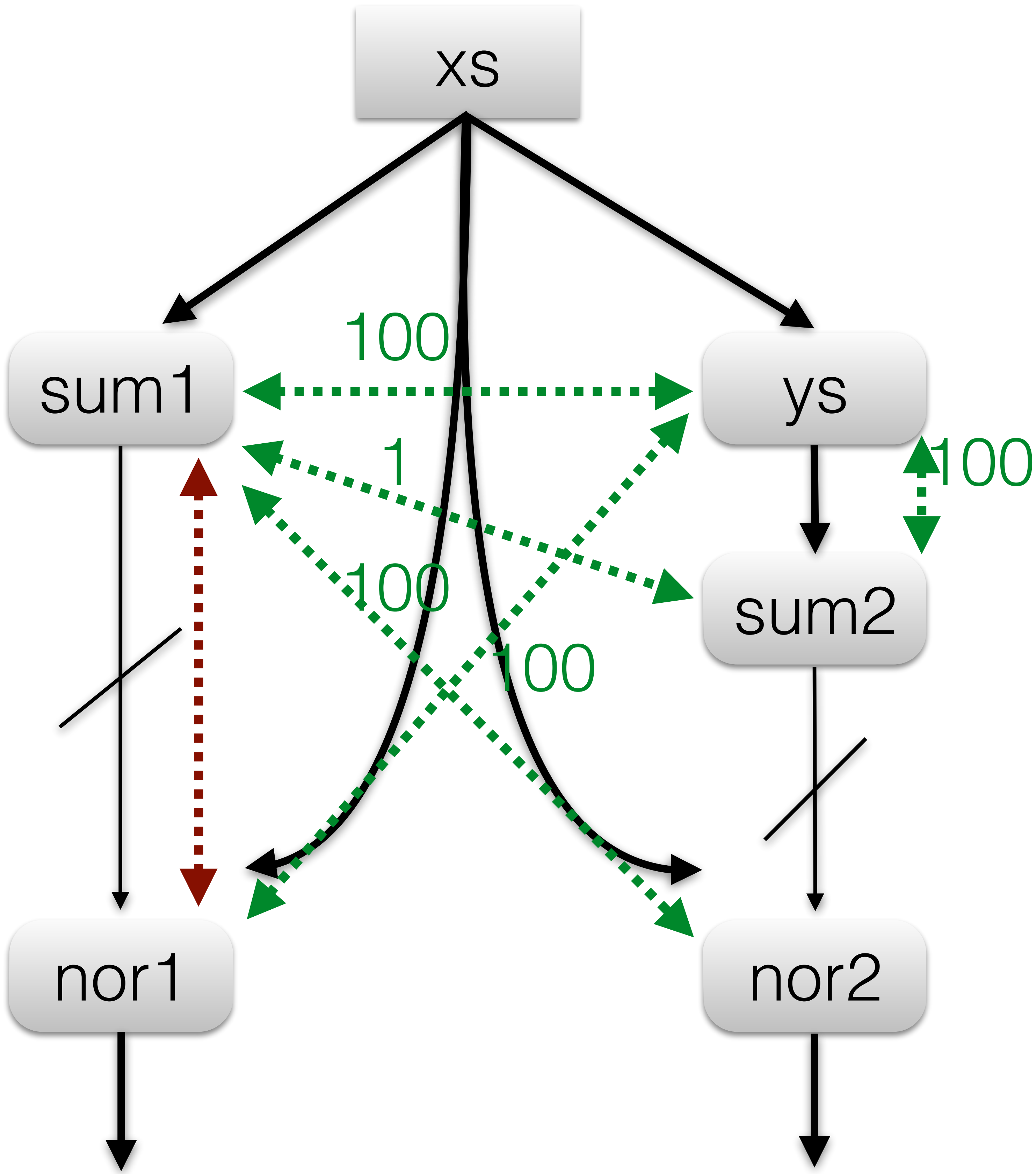

xs
100
sum1
ys
1
sum2
nor1
nor2

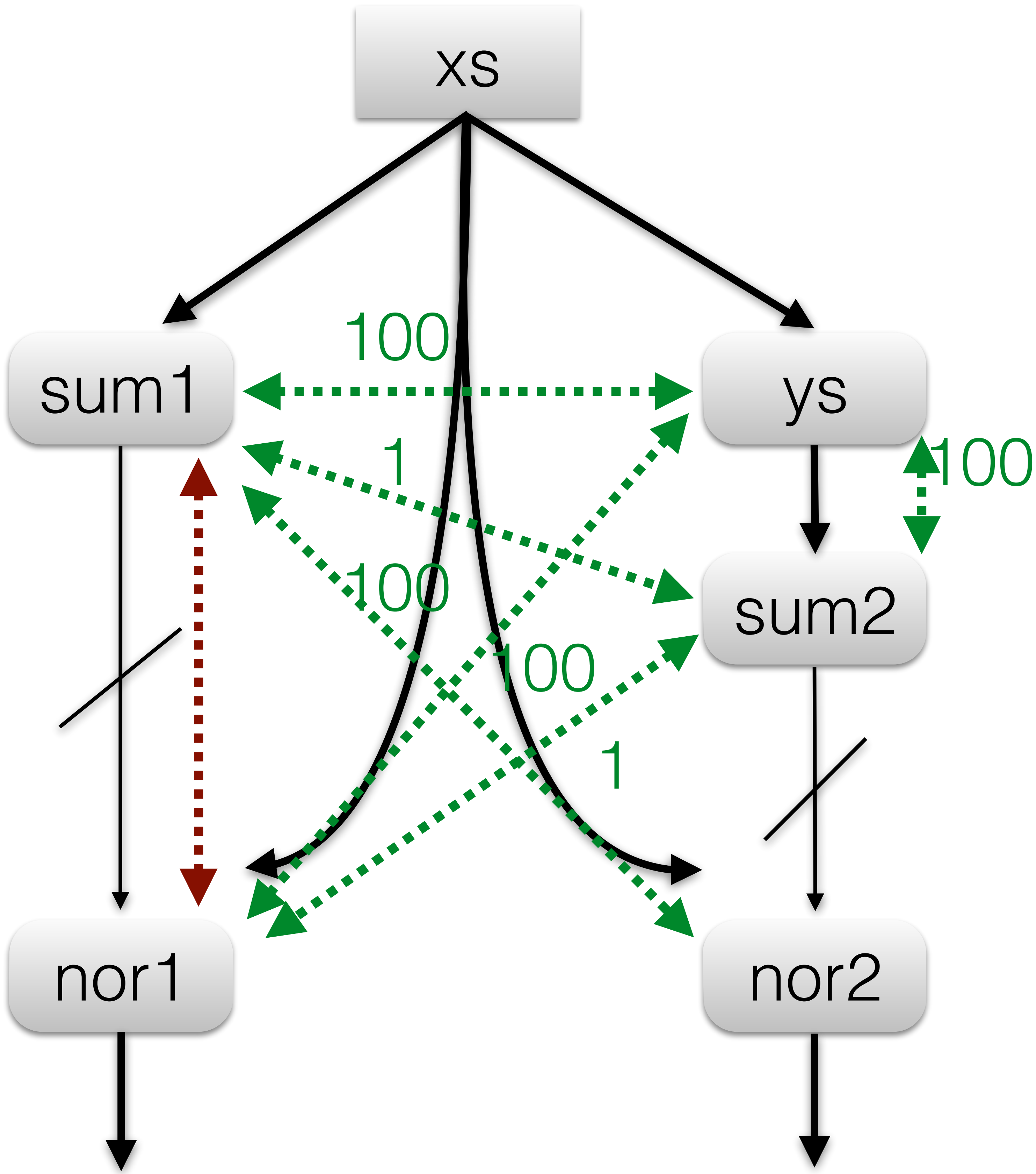

xs
sum1
100
ys
1
sum2
nor1
nor2



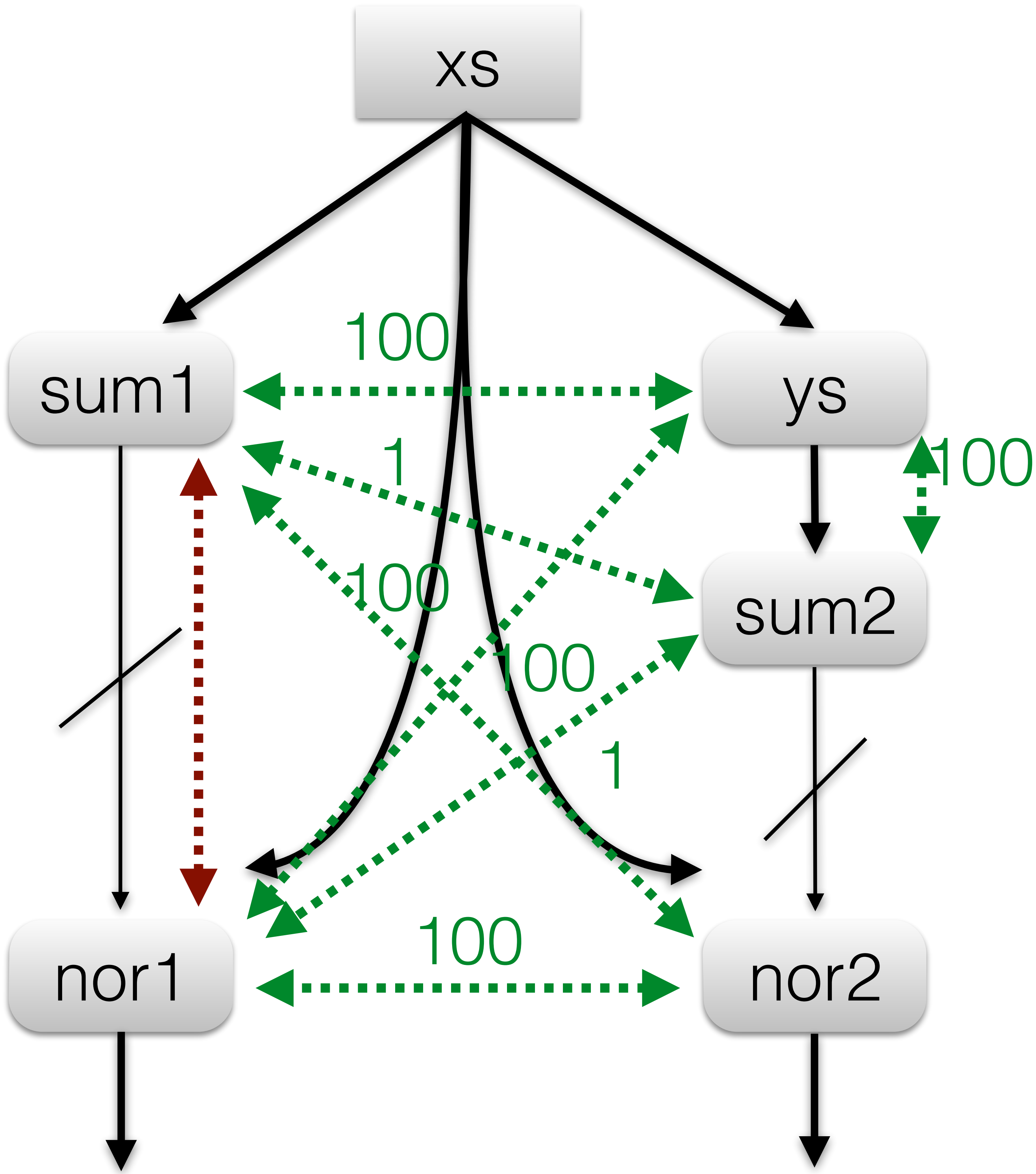
xs
100
sum1
ys
1
100
sum2
nor1
nor2


xs
100
sum1
ys
1
100
100
sum2
nor1
nor2


xs
sum1
ys
sum2
nor1
nor2
100
1
100
100
100



xs
sum1
ys
sum2
nor1
nor2
100
1
100
100
100
1

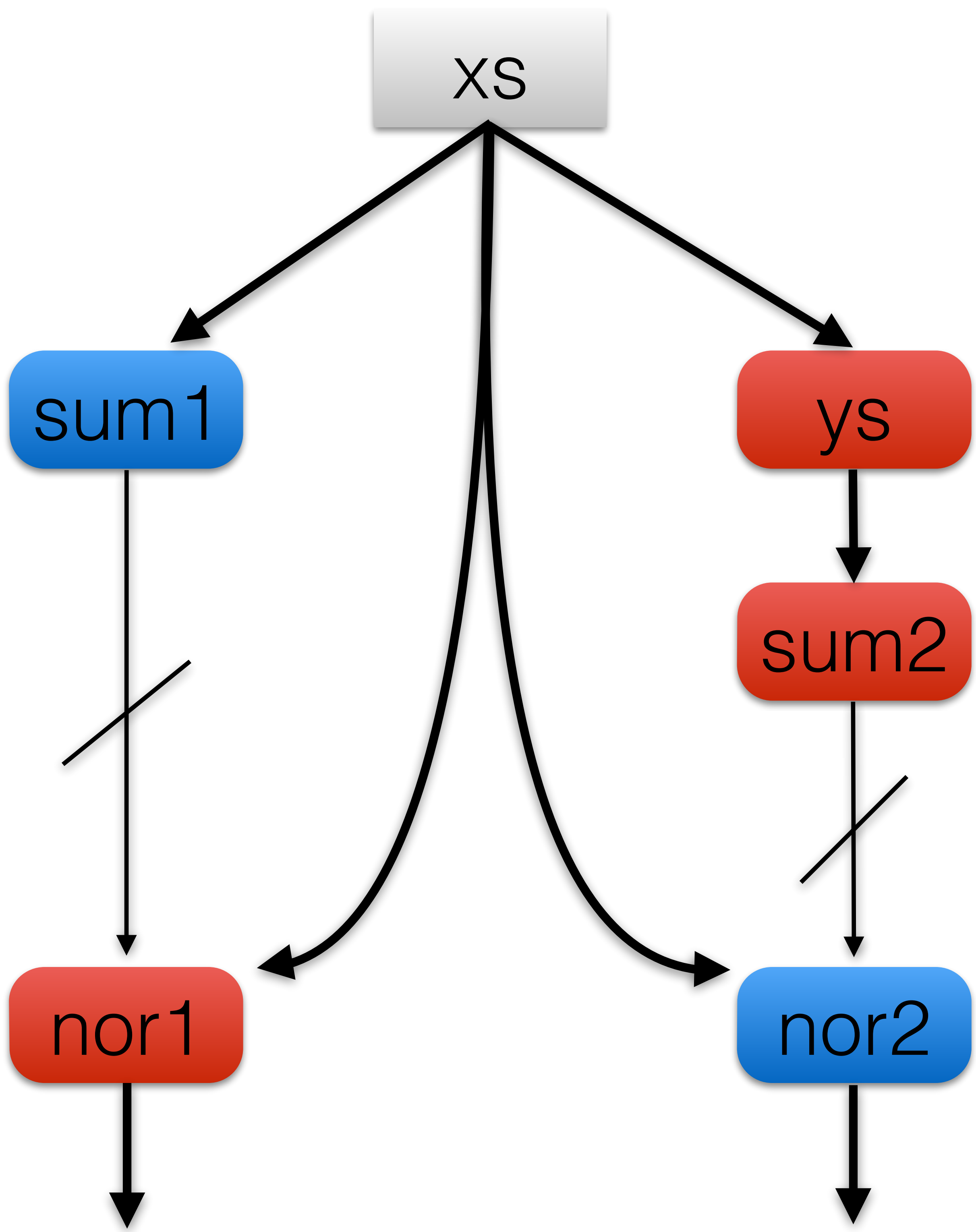


xs
sum1
ys
sum2
nor1
nor2
100
1
100
100
100
1
100

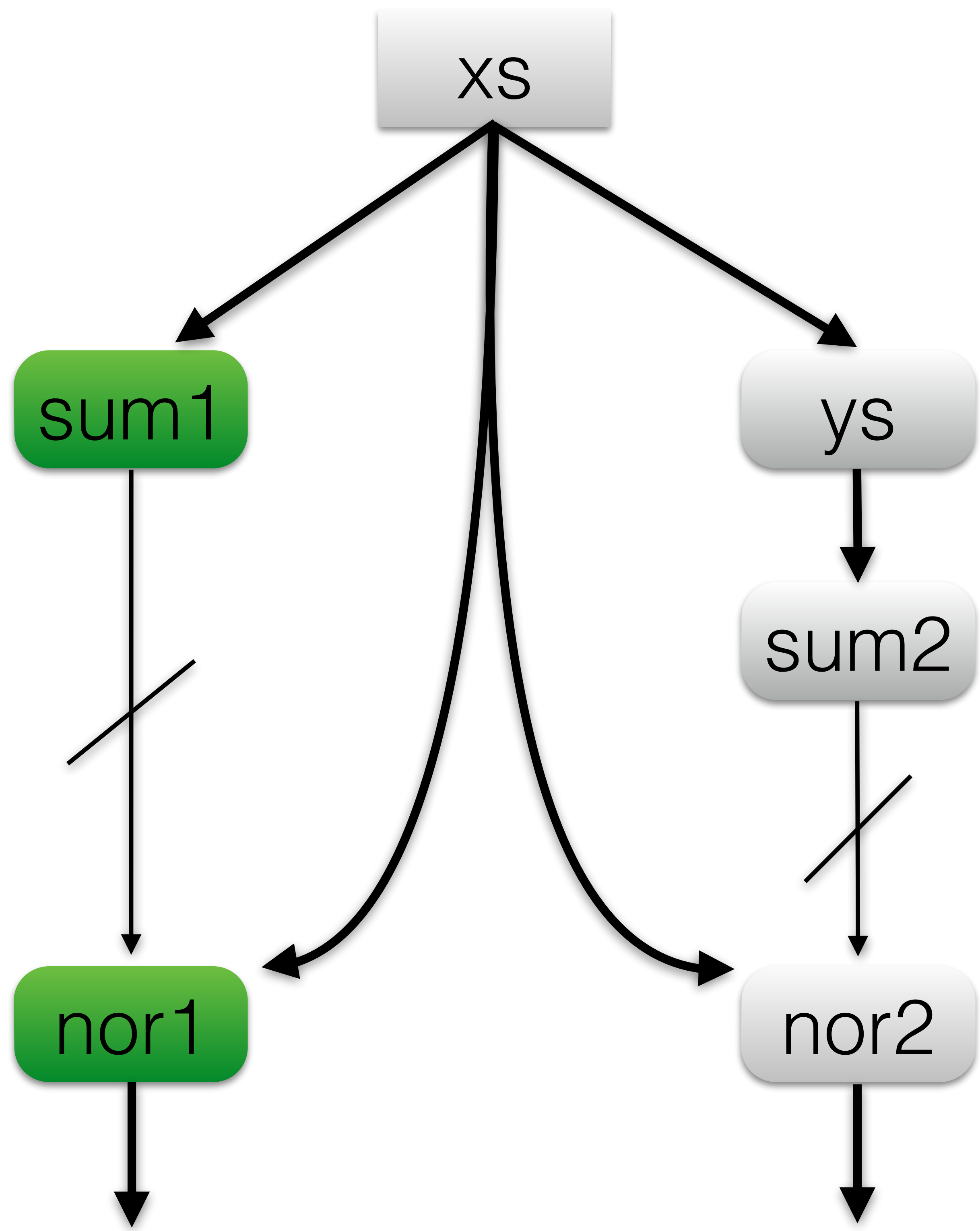
Objective function

$$
\begin{aligned}
\text{Minimise}\quad & 100f(\texttt{sum1}, \texttt{ys}) && + \quad 1f(\texttt{sum1}, \texttt{sum2}) \\
+\quad & 100f(\texttt{sum1}, \texttt{nor2}) && + \quad 100f(\texttt{ys}, \texttt{sum2}) \\
+\quad & 100f(\texttt{ys}, \texttt{nor1}) && + \quad 1f(\texttt{sum2}, \texttt{nor1}) \\
+\quad & 100f(\texttt{nor1}, \texttt{nor2}) &&
\end{aligned}
$$

Cyclic clusterings cannot be executed



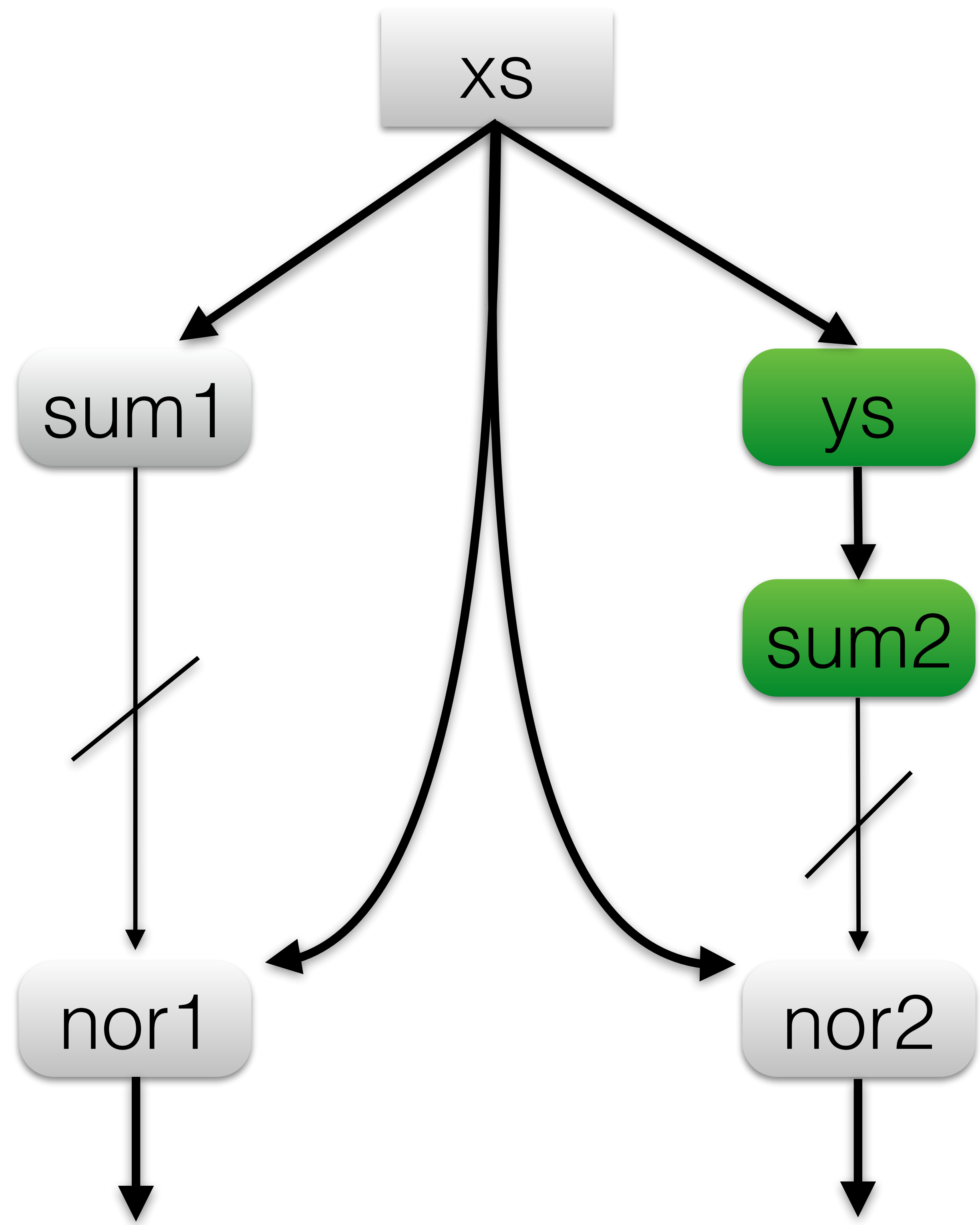
Non-fusible edge



Non-fusible edge

$o($`sum1`$) < o($`nor1`$)$

Fusible edge



Fusible edge

if $f(\texttt{ys}, \texttt{sum2}) = 0$

then $o(\texttt{ys}) = o(\texttt{sum2})$

else $o(\texttt{ys}) < o(\texttt{sum2})$

Fusible edge

$$1f(\texttt{ys},\texttt{sum2}) \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 100f(\texttt{ys}, \texttt{sum2})$$

Fusible edge - fused

$$1f(\texttt{ys},\texttt{sum2}) \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 100f(\texttt{ys}, \texttt{sum2})$$

$$0 \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 0$$

$$o(\texttt{sum2}) = o(\texttt{ys})$$

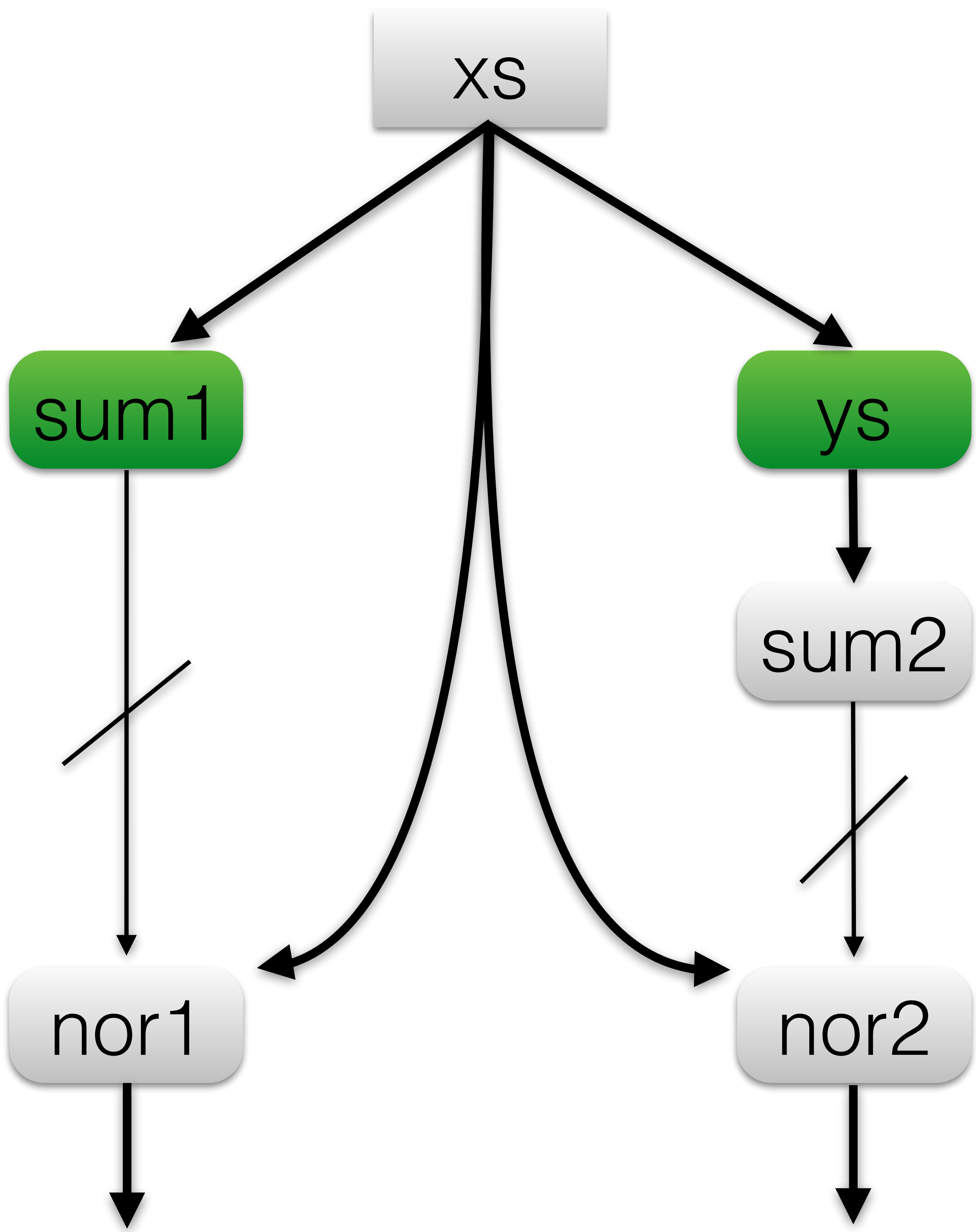
Fusible edge - unfused

$$1f(\texttt{ys},\texttt{sum2}) \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 100f(\texttt{ys}, \texttt{sum2})$$

$$1 \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 100$$

$$o(\texttt{sum2}) > o(\texttt{ys})$$

No edge



No edge

if $f(\texttt{sum1}, \texttt{ys}) = 0$

then $o(\texttt{sum1}) = o(\texttt{ys})$

No edge

$$-100f(\texttt{sum1}, \texttt{ys}) \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 100f(\texttt{sum1}, \texttt{ys})$$

No edge - fused

$$-100f(\texttt{sum1}, \texttt{ys}) \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 100f(\texttt{sum1}, \texttt{ys})$$

$$0 \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 0$$

$$o(\texttt{ys}) = o(\texttt{sum1})$$

No edge - unfused

$$-100f(\texttt{sum1}, \texttt{ys}) \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 100f(\texttt{sum1}, \texttt{ys})$$

$$-100 \qquad \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 100$$

All together

Minimise $100f(\texttt{sum1}, \texttt{ys}) + 1f(\texttt{sum1}, \texttt{sum2})$
$+ 100f(\texttt{sum1}, \texttt{nor2}) + 100f(\texttt{ys}, \texttt{sum2})$
$+ 100f(\texttt{ys}, \texttt{nor1}) + 1f(\texttt{sum2}, \texttt{nor1})$
$+ 100f(\texttt{nor1}, \texttt{nor2})$

Subject to

$$f(\texttt{sum1}, \texttt{ys}) \leq f(\texttt{sum1}, \texttt{sum2})$$

$$f(\texttt{sum2}, \texttt{ys}) \leq f(\texttt{sum1}, \texttt{sum2})$$

$$-100f(\texttt{sum1}, \texttt{ys}) \leq o(\texttt{ys}) - o(\texttt{sum1}) \leq 100f(\texttt{sum1}, \texttt{ys})$$

$$-100f(\texttt{sum1}, \texttt{sum2}) \leq o(\texttt{sum2}) - o(\texttt{sum1}) \leq 100f(\texttt{sum1}, \texttt{sum2})$$

$$1f(\texttt{ys}, \texttt{sum2}) \leq o(\texttt{sum2}) - o(\texttt{ys}) \leq 100f(\texttt{ys}, \texttt{sum2})$$

$$-100f(\texttt{nor1}, \texttt{nor2}) \leq o(\texttt{nor2}) - o(\texttt{nor1}) \leq 100f(\texttt{nor1}, \texttt{nor2})$$

$$o(\texttt{sum1}) < o(\texttt{nor1})$$

$$o(\texttt{sum2}) < o(\texttt{nor2})$$

Result clustering

$f(\texttt{sum1}, \texttt{ys}) = 0$

$f(\texttt{ys}, \texttt{sum2}) = 0$

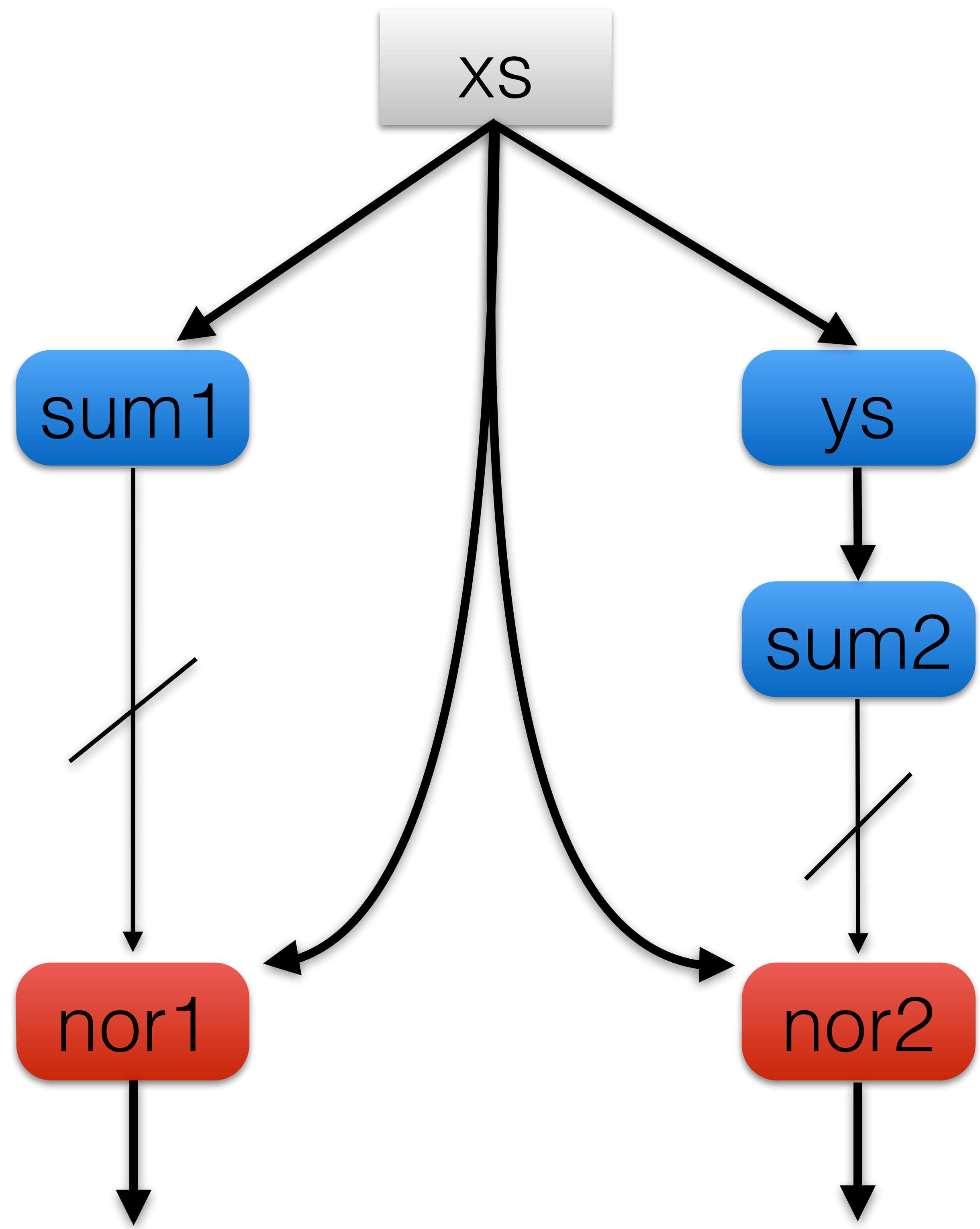
$f(\texttt{sum1}, \texttt{sum2}) = 0$

$f(\texttt{sum1}, \texttt{nor2}) = 1$

$f(\texttt{ys}, \texttt{nor1}) = 1$

$f(\texttt{sum2}, \texttt{nor1}) = 1$

$f(\texttt{nor1}, \texttt{nor2}) = 0$



In conclusion

- Integer linear programming isn't as scary as it sounds!
- We can fuse small (<10 combinator) programs in adequate time
- But we still need to look into large programs
- And we need to support more combinators
- Paper: http://www.cse.unsw.edu.au/~amosr/papers/robinson2014fusingfilters.pdf

Timing: small programs

- Quickhull, Normalize2, Closest points, Quad tree and other test cases
- GLPK and CPLEX both took < 100ms.

Timing: large program

- Randomly generated with 24 combinators
- GLPK (open source) took > 20min
- COIN/CBC (open source) took 90s
- CPLEX (commercial) took < 1s!

References

- Megiddo 1997: Optimal weighted loop fusion for parallel programs
- Darte 1999: On the complexity of loop fusion
- Lippmeier 2013: Data flow fusion with series expressions in Haskell

Differences from Megiddo

- With *combinators* instead of loops, we have more semantic information about the program.
- Which lets us recognise size-changing operations like filters, and fuse together.

Future work

- Currently only a few combinators: *map, map2, filter, fold, gather (bpermute), cross product*
- Need to support: *length, reverse, append, segmented fold, segmented map, segmented…*